On the Existence of Exhaustive Models in a Relational Feature Logic for Head-Driven Phrase Structure Grammar
- draft -

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Contents

1 RSRL 1
2 A Different Characterization of Exhaustive Models 7
3 Existence Proof for Exhaustive Models 12
References 26
1 RSRL

1 Definition. \( \text{Var} \) is a countably infinite set of symbols.

2 Definition. \( \Sigma \) is a signature iff

\[ \Sigma = \langle G, \sqsubseteq, S, A, F, R, AR \rangle, \]
\[ \langle G, \sqsubseteq \rangle \] is a finite partial order,
\[ S = \{ \sigma \in G \mid \text{for each } \sigma' \in G, \]
\[ \text{if } \sigma' \sqsubseteq \sigma \text{ then } \sigma = \sigma' \}, \]
\[ A \] is a set,
\[ F \] is a partial function from the Cartesian product of \( G \) and \( A \) to \( G \), and
\[ R \] is a finite set, and
\[ AR \] is a total function from \( R \) to \( \mathbb{N}^+ \).

Suppose \( S \) is a set. Throughout this paper we write \( S \) as an abbreviation for \( S \cup S^* \).

3 Definition. For each signature \( \Sigma = \langle G, \sqsubseteq, S, A, F, R, AR \rangle \), \( I \) is a \( \Sigma \) interpretation iff

\[ I = \langle U, S, A, R \rangle, \]
\[ U \] is a set of objects,
\[ S \] is a total function from \( U \) to \( S \),
\[ A \] is a total function from \( A \) to the set of partial functions from \( U \) to \( U \),
\[ \text{for each } \alpha \in A \text{ and each } u \in U, \]
\[ \text{if } A(\alpha)(u) \text{ is defined then } F(S(u), \alpha) \text{ is defined, } S(A(\alpha)(u)) \subseteq F(S(u), \alpha), \text{ and} \]
\[ \text{for each } \alpha \in A \text{ and each } u \in U, \]
\[ \text{if } F(S(u), \alpha) \text{ is defined then } A(\alpha)(u) \text{ is defined,} \]
\[ R \] is a total function from \( R \) to the powerset of \( U^* \), and
\[ \text{for each } \rho \in R, R(\rho) \subseteq U^{AR(\rho)}. \]

4 Definition. \( \langle \text{Chain}, \sqsubseteq^c \rangle \) is the smallest partial order such that
Chain = \{chain, echain, nechain\},
echain \sqsubseteq^c \text{ chain}, \text{ nechain} \sqsubseteq^c \text{ chain}.

5 Definition. For each signature \(\Sigma = \langle G, \sqsubseteq, S, A, F, R, AR\rangle\),
\[\hat{G} = G \cup \text{Chain} \cup \{\text{metatop}\},\]
\[\hat{\sqsubseteq} = \sqsubseteq \cup \sqsubseteq^c \cup \{\langle \sigma, \text{metatop} \rangle \mid \sigma \in \hat{G}\},\]
\[\hat{S} = S \cup \{\text{echain, nechain}\},\]
\[\hat{A} = A \cup \{\dagger, \triangleright\}.$

6 Definition. For each signature \(\Sigma = \langle G, \sqsubseteq, S, A, F, R, AR\rangle\), for each \(\Sigma\) interpretation \(I = \langle U, S, A, R\rangle\),
\[\hat{S} \text{ is the total function from } U \text{ to } \hat{S} \text{ such that}\]
for each \(u \in U\), \(\hat{S}(u) = S(u)\),
for each \(u_1 \in U\), \ldots, for each \(u_n \in U\),
\[\hat{S}(\langle u_1, \ldots, u_n \rangle) = \begin{cases} \text{echain} & \text{if } n = 0, \\ \text{nechain} & \text{if } n > 0 \end{cases}, \text{ and}\]
\(\hat{A} \) is the partial function from \(\hat{A}\) to the set of partial functions from \(U\) to \(U\) such that
for each \(\alpha \in A\), \(\hat{A}(\alpha) = A(\alpha)\), and
\(\hat{A}(\dagger)\) is the total function from \(U^+\) to \(U\) such that for each \(\langle u_0, \ldots, u_n \rangle \in U^+\),
\(\hat{A}(\dagger)(\langle u_0, \ldots, u_n \rangle) = u_0\), and
\(\hat{A}(\triangleright)\) is the total function from \(U^+\) to \(U^*\) such that for each \(\langle u_0, \ldots, u_n \rangle \in U^+\),
\(\hat{A}(\triangleright)(\langle u_0, \ldots, u_n \rangle) = \langle u_1, \ldots, u_n \rangle\).

7 Definition. For each signature \(\Sigma\), for each \(\Sigma\) interpretation \(I = \langle U, S, A, R\rangle\),
\[\text{Ass}_I = U^\text{Var} \text{ is the set of variable assignments in } I.\]

8 Definition. For each signature \(\Sigma = \langle G, \sqsubseteq, S, A, F, R, AR\rangle\), \(T^\Sigma\) is the smallest set such that
\[: \in T^\Sigma,\]
for each \(v \in \text{Var}\), \(v \in T^\Sigma\), and
for each \(\alpha \in \hat{A}\) and each \(\tau \in T^\Sigma\), \(\tau\alpha \in T^\Sigma\).

\[\text{Note that } \langle \hat{G}, \hat{\sqsubseteq} \rangle \text{ is a finite partial order.}\]
9 Definition. For each signature \( \Sigma = \langle \mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle \), for each \( \Sigma \) interpretation \( I = \langle U, S, A, R \rangle \), for each \( \text{ass} \in \text{Ass} \), \( T^\text{ass} \) is the total function from \( T^\Sigma \) to the set of partial functions from \( U \) to \( U \) such that for each \( u \in U \),

\[
T^\text{ass}_I(:)(u) \text{ is defined and } T^\text{ass}_I(:)(u) = u,
\]

for each \( v \in \text{Var} \), \( T^\text{ass}_I(v)(u) \) is defined and \( T^\text{ass}_I(v)(u) = \text{ass}(v) \),

for each \( \tau \in T^\Sigma \), for each \( \alpha \in \hat{A} \),

\[
T^\text{ass}_I(\tau \alpha)(u) \text{ is defined}
\]

if \( T^\text{ass}_I(\tau)(u) \) is defined and \( \hat{A}(\alpha)(T^\text{ass}_I(\tau)(u)) \) is defined, and

if \( T^\text{ass}_I(\tau \alpha)(u) \) is defined

then \( T^\text{ass}_I(\tau \alpha)(u) = \hat{A}(\alpha)(T^\text{ass}_I(\tau)(u)) \).

10 Definition. For each signature \( \Sigma = \langle \mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle \), \( D^\Sigma \) is the smallest set such that

for each \( \sigma \in \hat{G} \), for each \( \tau \in T^\Sigma \), \( \tau \sim \sigma \in D^\Sigma \),

for each \( \tau_1 \in T^\Sigma \), for each \( \tau_2 \in T^\Sigma \), \( \tau_1 \approx \tau_2 \in D^\Sigma \),

for each \( \rho \in \mathcal{R} \), for each \( x_1 \in \text{Var} \), \ldots, for each \( x_{\mathcal{AR}(\rho)} \in \text{Var} \), \( \rho(x_1, \ldots, x_{\mathcal{AR}(\rho)}) \in D^\Sigma \),

for each \( x \in \text{Var} \), for each \( \delta \in D^\Sigma \), \( \exists x \delta \in D^\Sigma \),

for each \( x \in \text{Var} \), for each \( \delta \in D^\Sigma \), \( \forall x \delta \in D^\Sigma \),

for each \( \delta \in D^\Sigma \), \( \neg \delta \in D^\Sigma \),

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \), \( [\delta_1 \land \delta_2] \in D^\Sigma \),

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \), \( [\delta_1 \lor \delta_2] \in D^\Sigma \),

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \), \( [\delta_1 \rightarrow \delta_2] \in D^\Sigma \), and

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \), \( [\delta_1 \leftrightarrow \delta_2] \in D^\Sigma \).

11 Definition. For each signature \( \Sigma = \langle \mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle \),

\[
FV(:) = \emptyset,
\]

for each \( v \in \text{Var} \), \( FV(v) = \{v\} \),

for each \( \tau \in T^\Sigma \), for each \( \alpha \in \hat{A} \), \( FV(\tau \alpha) = FV(\tau) \),

for each \( \tau \in T^\Sigma \), for each \( \sigma \in \hat{G} \), \( FV(\tau \sim \sigma) = FV(\tau) \),

for each \( \tau_1, \tau_2 \in T^\Sigma \), \( FV(\tau_1 \approx \tau_2) = FV(\tau_1) \cup FV(\tau_2) \),

for each \( \rho \in \mathcal{R} \), for each \( x_1, \ldots, x_{\mathcal{AR}(\rho)} \in \text{Var} \), \( FV(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)})) = \{x_1, \ldots, x_{\mathcal{AR}(\rho)}\} \),

for each \( \delta \in D^\Sigma \), for each \( v \in \text{Var} \), \( FV(\exists \delta \delta) = FV(\delta) \setminus \{v\} \),

for each \( \delta \in D^\Sigma \), for each \( v \in \text{Var} \), \( FV(\forall \delta \delta) = FV(\delta) \setminus \{v\} \),
for each $\delta \in \mathcal{D}^\Sigma$, $FV(\neg \delta) = FV(\delta)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, for each $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \land \delta_2) = FV(\delta_1) \cup FV(\delta_2)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, for each $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \lor \delta_2) = FV(\delta_1) \cup FV(\delta_2)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, for each $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \rightarrow \delta_2) = FV(\delta_1) \cup FV(\delta_2)$, and
for each $\delta_1 \in \mathcal{D}^\Sigma$, for each $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \leftrightarrow \delta_2) = FV(\delta_1) \cup FV(\delta_2)$.

**12 Definition.** For each signature $\Sigma = \langle \mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, and for each $u \in U$,

$$Co_I^u = \left\{ u' \in U \mid \begin{array}{l}
\text{for some } \text{ass} \in \text{Ass}_I, \\
\text{for some } \pi \in \mathcal{A}^*, \\
T^{\text{ass}}(\pi)(u) \text{ is defined, and} \\
u' = T^{\text{ass}}(\pi)(u)
\end{array} \right\}.$$ 

We call $Co_I^u$ the set of components of $u$ in $I$.

**13 Definition.** For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $\text{ass} \in \text{Ass}_I$, for each $v \in \text{Var}$, for each $w \in \text{Var}$, for each $u \in U$,

$$\text{ass}_{\text{ass}}^u(w) = \left\{ \begin{array}{l}
u = w \text{ if } v = w \\
\text{ass}(w) \text{ otherwise.}
\end{array} \right.$$ 

**14 Definition.** For each signature $\Sigma = \langle \mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $\text{ass} \in \text{Ass}_I$, $D^{\text{ass}}_I$ is the total function from $\mathcal{D}^\Sigma$ to the powerset of $U$ such that

for each $\tau \in \mathcal{T}^\Sigma$, for each $\sigma \in \mathcal{G}$,

$$D^{\text{ass}}_I(\tau \sim \sigma) = \left\{ u \in U \mid \begin{array}{l}
T^{\text{ass}}(\tau)(u) \text{ is defined, and} \\
S(T^{\text{ass}}(\tau)(u)) \supseteq \sigma
\end{array} \right\},$$

for each $\tau_1 \in \mathcal{T}^\Sigma$, for each $\tau_2 \in \mathcal{T}^\Sigma$,

$$D^{\text{ass}}_I(\tau_1 \approx \tau_2) = \left\{ u \in U \mid \begin{array}{l}
T^{\text{ass}}(\tau_1)(u) \text{ is defined,} \\
T^{\text{ass}}(\tau_2)(u) \text{ is defined, and} \\
T^{\text{ass}}(\tau_1)(u) = T^{\text{ass}}(\tau_2)(u)
\end{array} \right\},$$

for each $\rho \in \mathcal{R}$, for each $x_1 \in \text{Var}$, $\ldots$, for each $x_{\mathcal{AR}(\rho)} \in \text{Var}$,

$$D^{\text{ass}}_I(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)})) = \left\{ u \in U \mid \langle \text{ass}(x_1), \ldots, \text{ass}(x_{\mathcal{AR}(\rho)}) \rangle \in R(\rho) \right\},$$

for each $v \in \text{Var}$, for each $\delta \in \mathcal{D}^\Sigma$,
\[ D_I^{\text{ass}}(\exists v \, \delta) = \left\{ u \in U \mid \text{for some } u' \in \overline{\text{Co}_I^\delta}, \ u \in D_I^{\text{ass}}(\delta) \right\}, \]

for each \( v \in \text{Var} \), for each \( \delta \in D^\Sigma \),

\[ D_I^{\text{ass}}(\forall v \, \delta) = \left\{ u \in U \mid \text{for each } u' \in \overline{\text{Co}_I^\delta}, \ u \in D_I^{\text{ass}}(\delta) \right\}, \]

for each \( \delta \in D^\Sigma \),

\[ D_I^{\text{ass}}(\neg \delta) = U \setminus D_I^{\text{ass}}(\delta), \]

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \),

\[ D_I^{\text{ass}}([\delta_1 \land \delta_2]) = D_I^{\text{ass}}(\delta_1) \cap D_I^{\text{ass}}(\delta_2), \]

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \),

\[ D_I^{\text{ass}}([\delta_1 \lor \delta_2]) = D_I^{\text{ass}}(\delta_1) \cup D_I^{\text{ass}}(\delta_2), \]

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \),

\[ D_I^{\text{ass}}([\delta_1 \rightarrow \delta_2]) = (U \setminus D_I^{\text{ass}}(\delta)) \cup D_I^{\text{ass}}(\delta_2), \] and

for each \( \delta_1 \in D^\Sigma \), for each \( \delta_2 \in D^\Sigma \),

\[ D_I^{\text{ass}}([\delta_1 \leftrightarrow \delta_2]) = (U \setminus D_I^{\text{ass}}(\delta_1)) \cap (U \setminus D_I^{\text{ass}}(\delta_2)) \cup (D_I^{\text{ass}}(\delta_1) \cap D_I^{\text{ass}}(\delta_2)). \]

15 Proposition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I \), for each \( \text{ass}_1 \in \text{Ass}_I \),

for each \( \text{ass}_2 \in \text{Ass}_I \),

for each \( \tau \in T^\Sigma \),

if for each \( v \in \text{FV}(\tau) \), \( \text{ass}_1(v) = \text{ass}_2(v) \) then \( T_I^{\text{ass}_1}(\tau) = T_I^{\text{ass}_2}(\tau) \), and

for each \( \delta \in D^\Sigma \),

if for each \( v \in \text{FV}(\delta) \), \( \text{ass}_1(v) = \text{ass}_2(v) \) then \( D_I^{\text{ass}_1}(\delta) = D_I^{\text{ass}_2}(\delta) \).

16 Definition. For each signature \( \Sigma \),

\[ D^\Sigma_0 = \{ \delta \in D^\Sigma \mid \text{FV}(\delta) = \emptyset \}. \]

17 Corollary. For each signature \( \Sigma \), for each \( \delta \in D^\Sigma_0 \), for each \( \Sigma \) interpretation \( I \), for each \( \text{ass}_1 \in \text{Ass}_I \), for each \( \text{ass}_2 \in \text{Ass}_2 \),
18 Definition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, $D_I$ is the total function from $D^\Sigma_0$ to the powerset of $U$ such that for each $\delta \in D^\Sigma_0$,
\[ D_I(\delta) = \left\{ u \in U \mid \text{for each } \text{ass} \in Ass_I, \ u \in D^\text{ass}_I(\delta) \right\}. \]

19 Definition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, $\Theta_I$ is the total function from the powerset of $D^\Sigma_0$ to the powerset of $U$ such that for each $\theta \subseteq D^\Sigma_0$,
\[ \Theta_I(\theta) = \left\{ u \in U \mid \text{for each } \delta \in \theta, \ u \in D_I(\delta) \right\}. \]

20 Definition. $\Gamma$ is a grammar iff
\begin{enumerate}
\item $\Gamma$ is a pair $\langle \Sigma, \theta \rangle$,
\item $\Sigma$ is a signature, and
\item $\theta \subseteq D^\Sigma_0$.
\end{enumerate}

21 Definition. For each grammar $\Gamma = \langle \Sigma, \theta \rangle$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$,
\[ I \text{ is a } \Gamma \text{ model iff } \Theta_I(\theta) = U. \]

22 Definition. For each grammar $\Gamma = \langle \Sigma, \theta \rangle$, for each $\Sigma$ interpretation $I$,
\[ I \text{ is an exhaustive } \Gamma \text{ model iff } \]
\[ I \text{ is a } \Gamma \text{ model, and } \]
\[ \text{for each } \theta' \subseteq D^\Sigma_0, \text{ for each } \Sigma \text{ interpretation } I', \]
\[ \text{if } I' \text{ is a } \Gamma \text{ model and } \Theta_{I'}(\theta') \neq \emptyset, \]
\[ \text{then } \Theta_I(\theta') \neq \emptyset. \]
2 A Different Characterization of Exhaustive Models

Suppose that \( S \) is a set. We write \( \overline{S}^* \) for \((\overline{S})^*\) and \( S^{**} \) for \((S^*)^*\).

23 Definition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( u_1 \in U_1 \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( u_2 \in U_2 \),

\[ f \text{ is a congruence from } \langle u_1, I_1 \rangle \text{ to } \langle u_2, I_2 \rangle \text{ in } \Sigma \]

iff \( f \) is a bijection from \( \overline{Co}^I_{I_1} \) to \( \overline{Co}^I_{I_2} \),

for each \( u \in \overline{Co}^I_{I_1}, \overline{S}_1(u) = \overline{S}_2(f(u)), \)

for each \( \alpha \in \overline{A}, \) for each \( u \in \overline{Co}^I_{I_1}, \)

\[ \overline{A}_1(\alpha)(u) \text{ is defined iff } \overline{A}_2(\alpha)(f(u)) \text{ is defined, and if } \overline{A}_1(\alpha)(u) \text{ is defined then } f(\overline{A}_1(\alpha)(u)) = \overline{A}_2(\alpha)(f(u)), \]

for each \( \rho \in \overline{R}, \) for each \( o_1 \in \overline{Co}^I_{I_1}, \ldots, \) for each \( o_{AR(\rho)} \in \overline{Co}^I_{I_1}, \)

\[ \langle o_1, \ldots, o_{AR(\rho)} \rangle \in R_1(\rho) \text{ iff } f(o_1), \ldots, f(o_{AR(\rho)}) \in R_2(\rho), \text{ and } f(u_1) = u_2. \]

A first object in a first interpretation and a second object in a second interpretation are congruent iff there is a species, attribute and relation preserving bijection from the components of the first object in the first interpretation to the components of the second object in the second interpretation such that the bijection maps the first object to the second object.

24 Definition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( u_1 \in U_1 \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( u_2 \in U_2 \),

\[ \langle u_1, I_1 \rangle \text{ and } \langle u_2, I_2 \rangle \text{ are congruent in } \Sigma \]

iff for some \( f, \) \( f \) is a congruence from \( \langle u_1, I_1 \rangle \) to \( \langle u_2, I_2 \rangle \) in \( \Sigma \).

25 Definition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( u_1 \in U_1 \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( u_2 \in U_2 \),

\[ \langle u_1, I_1 \rangle \text{ and } \langle u_2, I_2 \rangle \text{ are indiscernible in } \Sigma \]

iff for each \( \delta \in D_0^\Sigma, u_1 \in D_{I_1}(\delta) \text{ iff } u_2 \in D_{I_2}(\delta). \)

We use a standard definition of functional composition:

26 Definition. For each set \( U \), for each set \( V \), for each set \( W \), for each total function \( f \) from \( U \) to \( V \), for each total function \( g \) from \( V \) to \( W \),

\[ g \circ f \text{ is the function from } U \text{ to } W \text{ such that, for each } u \in U, \]

\[ g \circ f(u) = g(f(u)). \]
By proposition 15 the following definition is well-made.

27 Definition. For each signature $\Sigma = (G, \sqsubseteq, S, A, F, R, AR)$, for each $\Sigma$ interpretation $I = (U, S, A, R)$, $T_I$ is the partial function from the Cartesian product of $\overline{AR}$ and $U$ to $U$ such that

for each $\pi \in A^*$, for each $u \in U$,

$$T_I(\pi, u) \text{ is defined iff for some } \text{ass} \in \text{Ass}_I, \ T_I^{\text{ass}}(\pi)(u) \text{ is defined, and}$$

$$\text{if } T_I(\pi, u) \text{ is defined then for some } \text{ass} \in \text{Ass}_I, \ T_I(\pi, u) = T_I^{\text{ass}}(\pi)(u),$$

for each $\langle \pi_1, \ldots, \pi_n \rangle \in A^{**}$, for each $u \in U$,

$$T_I(\langle \pi_1, \ldots, \pi_n \rangle, u) \text{ is defined}$$

$$\text{iff } T_I(\pi_1, u) \text{ is defined, \ldots, } T_I(\pi_n, u) \text{ is defined, and}$$

$$\text{if } T_I(\langle \pi_1, \ldots, \pi_n \rangle, u) \text{ is defined then } T_I(\langle \pi_1, \ldots, \pi_n \rangle, u) = \langle T_I(\pi_1, u), \ldots, T_I(\pi_n, u) \rangle.$$  

28 Proposition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I_1 = (U_1, S_1, A_1, R_1)$, for each $o_1 \in U_1$, for each $\Sigma$ interpretation $I_2 = (U_2, S_2, A_2, R_2)$, for each $o_2 \in U_2$,

$$\langle o_1, I_1 \rangle \text{ and } \langle o_2, I_2 \rangle \text{ are congruent in } \Sigma \text{ iff } \langle o_1, I_1 \rangle \text{ and } \langle o_2, I_2 \rangle \text{ are indiscernible in } \Sigma.$$  

Proof. Firstly, for each signature $\Sigma$, for each $\Sigma$ interpretation $I_1 = (U_1, S_1, A_1, R_1)$, for each $o_1 \in U_1$, for each $\Sigma$ interpretation $I_2 = (U_2, S_2, A_2, R_2)$, for each $o_2 \in U_2$, for each congruence $f$ from $\langle o_1, I_1 \rangle$ to $\langle o_2, I_2 \rangle$,

for each $\tau \in T^\Sigma$, for each total function $\text{ass}$ from $\text{Var}$ to $\overline{\text{Co}}_{I_1}^\varnothing$,

$$T_{I_1}^{\text{ass}}(\tau)(o_1) \text{ is defined iff } T_{I_2}^{f^{\text{ass}}}(\tau)(o_2) \text{ is defined, and}$$

$$\text{if } T_{I_1}^{\text{ass}}(\tau)(o_1) \text{ is defined then } f(T_{I_1}^{\text{ass}}(\tau)(o_1)) = T_{I_2}^{f^{\text{ass}}}(\tau)(o_2), \text{ and}$$

by induction on the length of $\tau$

for each $\delta \in D^\Sigma$, for each total function $\text{ass}$ from $\text{Var}$ to $\overline{\text{Co}}_{I_1}^\varnothing$,

$$o_1 \in D_{I_1}^{\text{ass}}(\delta) \text{ iff } o_2 \in D_{I_2}^{f^{\text{ass}}}(\delta), \text{ by induction on the complexity of } \delta$$

(since, for each $o \in U$, $f \circ (\text{ass}^\varnothing) = (f \circ \text{ass})^{f^{\varnothing}}$)

Thus, for each signature $\Sigma$, for each $\Sigma$ interpretation $I_1 = (U_1, S_1, A_1, R_1)$, for each $o_1 \in U_1$, for each $\Sigma$ interpretation $I_2 = (U_2, S_2, A_2, R_2)$, for each $o_2 \in U_2$,

$$\langle o_1, I_1 \rangle \text{ and } \langle o_2, I_2 \rangle \text{ are congruent in } \Sigma$$

$$\implies \text{ for some congruence } f, f \text{ is a congruence from } \langle o_1, I_1 \rangle \text{ to } \langle o_2, I_2 \rangle \text{ in } \Sigma$$

$$\implies \text{ for each } \delta \in D^\Sigma_0,$$
$o_1 \in D_{I_1}(\delta)$
\iff for each \( \text{ass} \in \text{Ass}_{I_1}, o_1 \in D^{\text{ass}}_{I_1}(\delta) \)
\iff for some \( \text{ass} \in \text{Ass}_{I_1}, 
\begin{align*}
o_1 &\in D^{\text{ass}}_{I_1}(\delta) \text{ and, for each } v \in \text{Var}, \text{ass}(v) \in \text{Co}^{\text{ass}}_{I_1} \quad \text{by corollary 17}
\end{align*}
\iff for some \( \text{ass} \in \text{Ass}_{I_2}, 
\begin{align*}
o_2 &\in D^{\text{ass}}_{I_2}(\delta) \text{ and, for each } v \in \text{Var}, \text{ass}(v) \in \text{Co}^{\text{ass}}_{I_2}
\end{align*}
\iff for each \( \text{ass} \in \text{Ass}_{I_2}, o_2 \in D^{\text{ass}}_{I_2}(\delta) 
\iff o_2 \in D_{I_2}(\delta)
\implies \langle o_1, I_1 \rangle \text{ and } \langle o_2, I_2 \rangle \text{ are indiscernible in } \Sigma.

Secondly, suppose \( \Sigma = (\mathcal{G}, \subseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR}) \) is a signature, \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \) is a \( \Sigma \) interpretation, \( o_1 \in U_1 \), \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \) is a \( \Sigma \) interpretation, \( o_2 \in U_2 \), and \( \langle o_1, I_1 \rangle \) and \( \langle o_2, I_2 \rangle \) are indiscernible in \( \Sigma \).

Firstly, for each \( \pi \in \mathcal{A}^* \), \( T_{I_1}(\pi, o_1) \) is defined iff \( T_{I_2}(\pi, o_2) \) is defined.

by induction on the length of \( \pi \)

Secondly, let 
\[
f = \left\{ \langle o_1', o_2' \rangle \in \text{Co}^{\alpha_1}_{I_1} \times \text{Co}^{\alpha_2}_{I_2} \mid 
\begin{align*}
T_{I_1}(\pi, o_1) &\text{ is defined,} \\
T_{I_2}(\pi, o_2) &\text{ is defined,} \\
o_1' &= T_{I_1}(\pi, o_1), \text{ and} \\
o_2' &= T_{I_2}(\pi, o_2) \end{align*}
\right\}.
\]

\( f \) is a bijection from \( \text{Co}^{\alpha_1}_{I_1} \) to \( \text{Co}^{\alpha_2}_{I_2} \).

Let \( \overline{f} \) be the total function from \( \text{Co}^{\alpha_1}_{I_1} \) to \( \text{Co}^{\alpha_2}_{I_2} \) such that,

\[
\begin{align*}
\text{for each } u \in \text{Co}^{\alpha_1}_{I_1}, \overline{f}(u) &= f(u), \text{ and} \\
\text{for each } \langle u_1, \ldots, u_n \rangle \in (\text{Co}^{\alpha_1}_{I_1})^*, \overline{f}(\langle u_1, \ldots, u_n \rangle) &= \langle f(u_1), \ldots, f(u_n) \rangle.
\end{align*}
\]

Clearly, \( \overline{f} \) is a bijection.

Thirdly, for each \( u \in \text{Co}^{\alpha_1}_{I_1}, S_1(u) = S_2(f(u)) \). Thus, for each \( u \in \text{Co}^{\alpha_1}_{I_1}, \overline{S}_1(u) = \overline{S}_2(\overline{f}(u)).
\]

Fourthly, for each \( \alpha \in \mathcal{A} \), for each \( u \in \text{Co}^{\alpha_1}_{I_1}, 
\begin{align*}
A_1(\alpha)(u) &\text{ is defined iff } A_2(\alpha)(f(u)) \text{ is defined, and} \\
&\text{if } A_1(\alpha)(u) \text{ is defined then } f(A_1(\alpha)(u)) = A_2(\alpha)(f(u)).
\end{align*}

Thus, for each \( \alpha \in \mathcal{A} \), for each \( u \in \text{Co}^{\alpha_1}_{I_1}, 
\begin{align*}
\overline{A}_1(\alpha)(u) &\text{ is defined iff } \overline{A}_2(\alpha)(\overline{f}(u)) \text{ is defined, and} \\
&\text{if } \overline{A}_1(\alpha)(u) \text{ is defined then } \overline{f}(\overline{A}_1(\alpha)(u)) = \overline{A}_2(\alpha)(\overline{f}(u)).
\end{align*}

Fifthly, for each \( \rho \in \mathcal{R} \), for each \( u_1 \in \text{Co}_I^\rho \), \( \ldots \), for each \( u_n \in \text{Co}_I^\rho \),

\[
\langle u_1, \ldots, u_n \rangle \in R_{I_1}(\rho),
\]

for some \( \pi_1 \in \mathcal{A}^\rho \), \( \ldots \), for some \( \pi_n \in \mathcal{A}^\rho \),

\[
u_1 = T_{I_1}(\pi_1, o_1), \ldots, u_n = T_{I_1}(\pi_n, o_1), \text{ and } \langle u_1, \ldots, u_n \rangle \in R_{I_1}(\rho)
\]

for some \( \pi_1 \in \mathcal{A}^\rho \), \( \ldots \), for some \( \pi_n \in \mathcal{A}^\rho \),

\[
u_1 = T_{I_1}(\pi_1, o_1), \ldots, u_n = T_{I_1}(\pi_n, o_1), \text{ and } o_1 \in D_{I_1}(\exists x_1 \ldots \exists x_n (\rho(x_1, \ldots, x_n) \land x_1 \approx : \pi_1 \land \ldots \land x_n \approx : \pi_n))^2
\]

for some \( \pi_1 \in \mathcal{A}^\rho \), \( \ldots \), for some \( \pi_n \in \mathcal{A}^\rho \),

\[
\overrightarrow{f}(u_1) = T_{I_2}(\pi_1, o_2), \ldots, \overrightarrow{f}(u_n) = T_{I_2}(\pi_n, o_2), \text{ and } o_2 \in D_{I_2}(\exists x_1 \ldots \exists x_n (\rho(x_1, \ldots, x_n) \land x_1 \approx : \pi_1 \land \ldots \land x_n \approx : \pi_n))
\]

for some \( \pi_1 \in \mathcal{A}^\rho \), \( \ldots \), for some \( \pi_n \in \mathcal{A}^\rho \),

\[
\overrightarrow{f}(u_1) = T_{I_2}(\pi_1, o_2), \ldots, \overrightarrow{f}(u_n) = T_{I_2}(\pi_n, o_2), \text{ and } \langle \overrightarrow{f}(u_1), \ldots, \overrightarrow{f}(u_n) \rangle \in R_{I_2}(\rho)
\]

\[
\iff \langle \overrightarrow{f}(u_1), \ldots, \overrightarrow{f}(u_n) \rangle \in R_{I_2}(\rho).
\]

Finally, \( \overrightarrow{f}(o_1) = o_2 \).

Therefore, \( \overrightarrow{f} \) is a congruence from \( \langle o_1, I_1 \rangle \) to \( \langle o_2, I_2 \rangle \) in \( \Sigma \). \( \langle o_1, I_1 \rangle \) and \( \langle o_2, I_2 \rangle \) are, thus, congruent in \( \Sigma \). \( \blacksquare \)

29 Definition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \),

\( I_1 \) simulates \( I_2 \) in \( \Sigma \)

iff for each \( u_2 \in U_2 \), for some \( u_1 \in U_1 \), \( \langle u_1, I_1 \rangle \) and \( \langle u_2, I_2 \rangle \) are congruent in \( \Sigma \).

A \( \Sigma \) interpretation \( I_1 \) simulates \( \Sigma \) interpretation \( I_2 \) just in case every object in \( I_2 \) has a congruent counterpart in \( I_1 \).

30 Proposition. For each signature \( \Sigma \),

for each \( \Sigma \) interpretation \( I \),

\( I \) simulates \( I \) in \( \Sigma \), and

for each \( \Sigma \) interpretation \( I_1 \), for each \( \Sigma \) interpretation \( I_2 \), for each \( \Sigma \) interpretation \( I_3 \),

if \( I_1 \) simulates \( I_2 \) in \( \Sigma \) and \( I_2 \) simulates \( I_3 \) in \( \Sigma \) then \( I_1 \) simulates \( I_3 \) in \( \Sigma \).

31 Proposition. For each signature \( \Sigma \), for each \( \theta \in \mathcal{D}_0^\Sigma \), for each \( \Sigma \) interpretation \( I \),

\(^2\)We write \( x \approx \langle \pi_1, \ldots, \pi_n \rangle \) as an abbreviation for \( x \vdash \approx : \pi_1 \land \ldots \land x \vdash \approx : \pi_{i+1} \land \ldots \land x \vdash \approx : \pi_n \sim \text{echain} \)
for each $\Sigma$ interpretation $I'$,

if $I'$ is a $(\Sigma, \theta)$ model then $I$ simulates $I'$ in $\Sigma$

iff for each $\theta' \subseteq D_0^{\Sigma}$, for each $\Sigma$ interpretation $I'$,

if $I'$ is a $(\Sigma, \theta)$ model and $\Theta_I(\theta') \neq \emptyset$ then $\Theta_I(\theta') \neq \emptyset$.

**Proof.** Firstly, for each signature $\Sigma$, for each $\theta \subseteq D_0^{\Sigma}$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$,

for each $\Sigma$ interpretation $I' = \langle U', S', A', R' \rangle$,

if $I'$ is a $(\Sigma, \theta)$ model then $I$ simulates $I'$

$\implies$ for each $\theta' \subseteq D_0^{\Sigma}$, for each $\Sigma$ interpretation $I' = \langle U', S', A', R' \rangle$,

$I'$ is a $(\Sigma, \theta)$ model and $\Theta_I(\theta') \neq \emptyset$

$\implies I$ simulates $I'$ in $\Sigma$ and, for some $u' \in U'$, $u' \in \Theta_I(\theta')$

$\implies$ for some $u \in U$, for some $u' \in U'$,

$\langle u, I \rangle$ and $\langle u', I' \rangle$ are congruent in $\Sigma$ and $u' \in \Theta_I(\theta')$

$\implies$ for some $u \in U$, for some $u' \in U'$,

$\langle u, I \rangle$ and $\langle u', I' \rangle$ are indiscernible in $\Sigma$ and $u' \in \Theta_I(\theta')$ by proposition 28

$\implies$ for some $u \in U$, $u \in \Theta_I(\theta')$

$\implies \Theta_I(\theta') \neq \emptyset$.

Secondly, for each signature $\Sigma$, for each $\theta \subseteq D_0^{\Sigma}$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$,

for each $\theta' \subseteq D_0^{\Sigma}$, for each $\Sigma$ interpretation $I'$,

if $I'$ is a $(\Sigma, \theta)$ model and $\Theta_I(\theta') \neq \emptyset$ then $\Theta_I(\theta') \neq \emptyset$

$\implies$ for each $\Sigma$ interpretation $I' = \langle U', S', A', R' \rangle$,

$I'$ is a $(\Sigma, \theta)$ model

$\implies$ for each $u'$,

$u' \in U'$

$\implies u' \in \Theta_{I'} \left\{ \delta \in D_0^{\Sigma} \mid u' \in D_{I'}(\delta) \right\}$

$\implies \Theta_{I'} \left\{ \delta \in D_0^{\Sigma} \mid u' \in D_{I'}(\delta) \right\} \neq \emptyset$

$\implies \Theta_I \left\{ \delta \in D_0^{\Sigma} \mid u' \in D_{I'}(\delta) \right\} \neq \emptyset$

$\implies$ for some $u \in U$, $u \in \Theta_I \left\{ \delta \in D_0^{\Sigma} \mid u' \in D_{I'}(\delta) \right\}$

$\implies$ for some $u \in U$, for each $\delta \in D_0^{\Sigma}$,

$u \in D_I(\delta)$

$\implies u \notin D_I(\neg \delta)$
3 Existence Proof for Exhaustive Models

In this section we show that, for each grammar, there exists an exhaustive model. For each grammar \( \langle \Sigma, \theta \rangle \), we construct a \( \Sigma \) interpretation which we call the canonical \( \Sigma \) interpretation of \( \theta \). We then show that the canonical \( \Sigma \) interpretation is an exhaustive model of the grammar \( \langle \Sigma, \theta \rangle \).

Suppose \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \) is a signature. We call each member of \( A^* \) a \( \Sigma \) path, and write \( \varepsilon \) for the empty path, the unique path of length zero.

33 Definition. For each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \),

\( \mu \) is a morph in \( \Sigma \)

iff \( \mu \) is a quadruple \( \langle \beta, \varrho, \lambda, \xi \rangle \),

\( \beta \subseteq A^* \),

\( \varepsilon \in \beta \),

for each \( \pi \in A^* \), for each \( \alpha \in A \),

if \( \pi \alpha \in \beta \) then \( \pi \in \beta \),

\( \varrho \) is an equivalence relation over \( \beta \),

for each \( \pi_1 \in A^* \), for each \( \pi_2 \in A^* \), for each \( \alpha \in A \),

\[ \Rightarrow \neg \delta \notin \left\{ \delta \in D_0^\Sigma \mid u' \in D_I(\delta) \right\} \]
\[ \Rightarrow u' \notin D_I(\neg \delta) \]
\[ \Rightarrow u' \in D_I(\delta), \text{ and} \]
\[ u' \in D_I(\delta) \]
\[ \Rightarrow \delta \in \left\{ \delta \in D_0^\Sigma \mid u' \in D_I(\delta) \right\} \]
\[ \Rightarrow u \in D_I(\delta) \]
\[ \Rightarrow \text{for some} \ u \in U, \langle u, I \rangle \text{ and } \langle u', I' \rangle \text{ are indiscernible in } \Sigma \]
\[ \Rightarrow \text{for some} \ u \in U, \langle u, I \rangle \text{ and } \langle u', I' \rangle \text{ are congruent in } \Sigma \] by proposition 28
\[ \Rightarrow I \text{ simulates } I' \text{ in } \Sigma. \]

32 Theorem. For each signature \( \Sigma \), for each \( \theta \subseteq D_0^\Sigma \), for each \( \Sigma \) interpretation \( I \),

\( I \) is an exhaustive \( \langle \Sigma, \theta \rangle \) model

iff \( I \) is a \( \langle \Sigma, \theta \rangle \) model, and for each \( \Sigma \) interpretation \( I' \),

if \( I' \) is a \( \langle \Sigma, \theta \rangle \) model then \( I \) simulates \( I' \) in \( \Sigma \).
if $\pi_1\alpha \in \beta$ and $\langle \pi_1, \pi_2 \rangle \in \varrho$ then $\langle \pi_1\alpha, \pi_2\alpha \rangle \in \varrho$.

$\lambda$ is a total function from $\beta$ to $S$.

for each $\pi_1 \in A^*$, for each $\pi_2 \in A^*$,

if $\langle \pi_1, \pi_2 \rangle \in \varrho$ then $\lambda(\pi_1) = \lambda(\pi_2)$.

for each $\pi \in A^*$, for each $\alpha \in A$,

if $\pi\alpha \in \beta$ then $\mathcal{F}(\lambda(\pi), \alpha)$ is defined and $\lambda(\pi\alpha) \subseteq \mathcal{F}(\lambda(\pi), \alpha)$,

for each $\pi \in A^*$, for each $\alpha \in A$,

if $\pi\alpha \in \beta$ and $\mathcal{F}(\lambda(\pi), \alpha)$ is defined then $\pi\alpha \in \beta$,

$\xi \subseteq R \times \overline{\beta}'$,

for each $\rho \in R$, for each $\pi_1 \in \overline{\beta}$, ..., for each $\pi_n \in \overline{\beta}$,

if $\langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi$ then $n = AR(\rho)$, and

for each $\rho \in R$, for each $\pi_1 \in \overline{\beta}$, ..., for each $\pi_n \in \overline{\beta}$, for each $\pi'_1 \in \overline{\beta}$, ..., for each $\pi'_n \in \overline{\beta}$,

if $\langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi$, and

for each $i \in \{1, \ldots, n\}$,

$\pi_i \in \beta$ and $\langle \pi_i, \pi_i' \rangle \in \varrho$,

or

for some $m \in \mathbb{N}$,

$\pi_i \in \beta^*$,

$\pi_i = \langle \pi_{i_1}, \ldots, \pi_{i_m} \rangle$,

$\pi_i' = \langle \pi_{i_1}', \ldots, \pi_{i_m}' \rangle$, and

$\langle \pi_{i_1}, \pi_{i_1}' \rangle \in \varrho$, ..., $\langle \pi_{i_m}, \pi_{i_m}' \rangle \in \varrho$,

then $\langle \rho, \pi'_1, \ldots, \pi'_n \rangle \in \xi$.

Suppose $\Sigma$ is a signature and $\mu = \langle \beta, \varrho, \lambda, \xi \rangle$ is a $\Sigma$ morph. We call $\beta$ the basis set in $\mu$, $\varrho$ the re-entrancy relation in $\mu$, $\lambda$ the label function in $\mu$, and $\xi$ the relation extension in $\mu$. We write $M_\Sigma$ for the set of $\Sigma$ morphs. Our $\Sigma$ morphs are a straightforward extension of abstract feature structures in the sense of (Moshier 1988).

34 Definition. For each signature $\Sigma = \langle G, \subseteq, S, A, \mathcal{F}, R, ARi \rangle$, for each $\pi \in A^*$, for each $\langle \pi_1, \ldots, \pi_n \rangle \in A^{**}$,

$\pi \langle \pi_1, \ldots, \pi_n \rangle$ is an abbreviatory notation for $\langle \pi_1, \ldots, \pi_1, \pi_{\pi_1}, \ldots, \pi_{\pi_n} \rangle$.

35 Definition. For each signature $\Sigma = \langle G, \subseteq, S, A, \mathcal{F}, R, ARi \rangle$, for each $\mu = \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma$, for each $\pi \in A^*$,

$\beta/\pi = \{ \pi' \in A^* \mid \pi_{\pi'} \in \beta \}$,

$\varrho/\pi = \{ \langle \pi_1, \pi_2 \rangle \in A^* \times A^* \mid \langle \pi_{\pi_1}, \pi_{\pi_2} \rangle \in \varrho \}$,

$\lambda/\pi = \{ \langle \pi', \sigma \rangle \in A^* \times S \mid \langle \pi_{\pi'}, \sigma \rangle \in \lambda \}$,

$\xi/\pi = \{ \langle \rho, \pi_1, \ldots, \pi_n \rangle \in R \times \overline{\beta/\pi} \mid \langle \rho, \pi_{\pi_1}, \ldots, \pi_{\pi_n} \rangle \in \xi \}$, and

$\mu/\pi = \langle \beta/\pi, \varrho/\pi, \lambda/\pi, \xi/\pi \rangle$. 
36 Proposition. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, \mathcal{AR} \rangle$, for each $\mu = \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\pi \in A^*$,

if $\pi \in \beta$ then $\mu/\pi \in \mathcal{M}_\Sigma$.

37 Definition. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, \mathcal{AR} \rangle$,

$I_\Sigma$ is the set of total functions from $\text{Var}$ to $\overline{A}^*$. Let $\Sigma$ be a signature. Then we call each member of $I_\Sigma$ a $\Sigma$ insertion.

38 Definition. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, \mathcal{AR} \rangle$,

$T_\Sigma$ is the smallest partial function from $\overline{A}^* \times \hat{A}$ to $\overline{A}^*$ such that

for each $\pi \in A^*$, for each $\alpha \in A$,

$T_\Sigma(\pi, \alpha) = \pi\alpha$,

for each $\langle \pi_0, \ldots, \pi_n \rangle \in A^{**}$,

$T_\Sigma(\langle \pi_0, \ldots, \pi_n \rangle, \dagger) = \pi_0$,

$T_\Sigma(\langle \pi_0, \ldots, \pi_n \rangle, \triangleright) = \langle \pi_1, \ldots, \pi_n \rangle$.

39 Definition. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, \mathcal{AR} \rangle$, for each $\iota \in I_\Sigma$,

$\Pi_\Sigma^\iota$ is the smallest partial function from $T^\Sigma$ to $\overline{A}^*$ such that

$\Pi_\Sigma^\iota(\cdot) = \varepsilon$,

for each $v \in \text{Var}$, $\Pi_\Sigma^\iota(v) = \iota(v)$, and

for each $\tau \in T^\Sigma$, for each $\alpha \in \hat{A}$, $\Pi_\Sigma^\iota(\tau\alpha) = T_\Sigma(\Pi_\Sigma^\iota(\tau), \alpha)$.

Suppose $\Sigma$ is a signature, and $\iota$ is a $\Sigma$ insertion. Then we call each $\Pi_\Sigma^\iota$ the path insertion function for $\iota$ in $\Sigma$.

40 Definition. For each signature $\Sigma$, for each $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$,

$\hat{\varrho}$ is the smallest subset of $\overline{\beta} \times \overline{\beta}$ such that

$\varrho \subseteq \hat{\varrho}$, and

for each $\pi_1, \ldots, \pi_n \in \beta$, for each $\pi'_1, \ldots, \pi'_n \in \beta$,

if $\langle \pi_1, \pi_1' \rangle \in \varrho$, \ldots, and $\langle \pi_n, \pi_n' \rangle \in \varrho$
then $\langle \langle \pi_1, \ldots, \pi_n \rangle, \langle \pi_1', \ldots, \pi_n' \rangle \rangle \in \hat{\varrho}$,

$\hat{\lambda}$ is the total function from $\overline{\beta}$ to $\hat{S}$ such that
for each \( \pi \in \beta \), \( \hat{\lambda}(\pi) = \lambda(\pi) \),
for each \( \pi_1 \in \beta \), ..., for each \( \pi_n \in \beta \),
\[
\hat{\lambda}(\langle \pi_1, \ldots, \pi_n \rangle) = \begin{cases} 
\text{echain} & \text{if } n = 0, \\
\text{nechain} & \text{if } n > 0.
\end{cases}
\]

41 Definition. For each signature \( \Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle \), for each \( \iota \in \iota_{\Sigma} \), \( \Delta^\iota_{\Sigma} \) is the total function from \( \mathcal{D}^\Sigma \) to \( \mathcal{M}_{\Sigma} \) such that

for each \( \tau \in \mathcal{T}^\Sigma \), for each \( \sigma \in \hat{\mathcal{G}} \),
\[
\Delta^\iota_{\Sigma}(\tau \sim \sigma) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_{\Sigma} \middle| \begin{array}{l}
\Pi^\iota_{\Sigma}(\tau) \text{ is defined, and,} \\
\text{for some } \sigma' \in \hat{\mathcal{S}}, \\
\langle \Pi^\iota_{\Sigma}(\tau), \sigma' \rangle \in \hat{\lambda} \text{ and } \sigma' \sqsubseteq \sigma
\end{array} \right\},
\]
for each \( \tau_1 \in \mathcal{T}^\Sigma \), for each \( \tau_2 \in \mathcal{T}^\Sigma \),
\[
\Delta^\iota_{\Sigma}(\tau_1 \approx \tau_2) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_{\Sigma} \middle| \begin{array}{l}
\Pi^\iota_{\Sigma}(\tau_1) \text{ is defined,} \\
\Pi^\iota_{\Sigma}(\tau_2) \text{ is defined, and} \\
\langle \Pi^\iota_{\Sigma}(\tau_1), \Pi^\iota_{\Sigma}(\tau_2) \rangle \in \hat{\iota}
\end{array} \right\},
\]
for each \( \rho \in \mathcal{R} \), for each \( v_1 \in \text{Var} \), ..., for each \( v_n \in \text{Var} \),
\[
\Delta^\iota_{\Sigma}(\rho(v_1, \ldots, v_n)) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_{\Sigma} \middle| \langle \rho, \iota(v_1), \ldots, \iota(v_n) \rangle \in \xi \right\},
\]
for each \( v \in \text{Var} \), for each \( \delta \in \mathcal{D}^\Sigma \),
\[
\Delta^\iota_{\Sigma}(\exists v \, \delta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_{\Sigma} \middle| \begin{array}{l}
\text{for some } \pi \in \overline{\beta}, \\
\langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\iota_{\Sigma}(\exists \pi)(\delta)
\end{array} \right\},
\]
for each \( v \in \text{Var} \), for each \( \delta \in \mathcal{D}^\Sigma \),
\[
\Delta^\iota_{\Sigma}(\forall v \, \delta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_{\Sigma} \middle| \begin{array}{l}
\text{for each } \pi \in \overline{\beta}, \\
\langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\iota_{\Sigma}(\forall \pi)(\delta)
\end{array} \right\},
\]
for each \( \delta \in \mathcal{D}^\Sigma \),
\[
\Delta^\iota_{\Sigma}(\neg \delta) = \mathcal{M}_{\Sigma} \setminus \Delta^\iota_{\Sigma}(\delta),
\]
for each \( \delta_1 \in \mathcal{D}^\Sigma \), for each \( \delta_2 \in \mathcal{D}^\Sigma \),
\[
\Delta^\iota_{\Sigma}([\delta_1 \land \delta_2]) = \Delta^\iota_{\Sigma}(\delta_1) \cap \Delta^\iota_{\Sigma}(\delta_2),
\]
for each \( \delta_1 \in \mathcal{D}^\Sigma \), for each \( \delta_2 \in \mathcal{D}^\Sigma \),
\[
\Delta^\iota_{\Sigma}([\delta_1 \lor \delta_2]) = \Delta^\iota_{\Sigma}(\delta_1) \cup \Delta^\iota_{\Sigma}(\delta_2),
\]
for each \( \delta_1 \in \mathcal{D}^\Sigma \), for each \( \delta_2 \in \mathcal{D}^\Sigma \).
\[\Delta^\Sigma_2([\delta_1 \rightarrow \delta_2]) = (\mathcal{M}_\Sigma \setminus \Delta^\Sigma_2(\delta_1)) \cup \Delta^\Sigma_2(\delta_2),\]

for each \(\delta_1 \in \mathcal{D}^\Sigma\), for each \(\delta_2 \in \mathcal{D}^\Sigma\),

\[\Delta^\Sigma_2([\delta_1 \leftarrow \delta_2]) = ((\mathcal{M}_\Sigma \setminus \Delta^\Sigma_2(\delta_1)) \cap (\mathcal{M}_\Sigma \setminus \Delta^\Sigma_2(\delta_2))) \cup (\Delta^\Sigma_2(\delta_1) \cap \Delta^\Sigma_2(\delta_2)).\]

Let \(\Sigma\) be a signature, \(\mu\) a \(\Sigma\) morph, \(\iota\) a \(\Sigma\) insertion and \(\delta\) a \(\Sigma\) description. We call \(\Delta^\Sigma_2\) the morph satisfaction function in \(\Sigma\), and say \(\mu\) satisfies \(\delta\) under \(\iota\) in \(\Sigma\) if and only if \(\Delta^\Sigma_2(\delta) = \mu\).

**42 Proposition.** For each signature \(\Sigma\), for each \(\iota_1 \in I_\Sigma\), for each \(\iota_2 \in I_\Sigma\),

for each \(\tau \in T^\Sigma\),

if for each \(v \in FV(\tau)\), \(\iota_1(v) = \iota_2(v)\)

then \(\Pi^\Sigma_1(\tau)\) is defined iff \(\Pi^\Sigma_1(\tau)\) is defined, and

if \(\Pi^\Sigma_1(\tau)\) is defined then \(\Pi^\Sigma_1(\tau) = \Pi^\Sigma_2(\tau)\), and

for each \(\delta \in \mathcal{D}^\Sigma\),

if for each \(v \in FV(\delta)\), \(\iota_1(v) = \iota_2(v)\)

then \(\Delta^\Sigma_1(\delta) = \Delta^\Sigma_2(\delta)\).

**Proof.** By induction on the length of \(\tau\) and the complexity of \(\delta\), respectively. \(\blacksquare\)

**43 Definition.** For each signature \(\Sigma\),

\(M_\Sigma\) is the total function from \(\text{Pow}(\mathcal{D}^\Sigma_0)\) to \(\text{Pow}(\mathcal{M}_\Sigma)\) such that for each \(\theta \subseteq \mathcal{D}^\Sigma_0\),

\[M_\Sigma(\theta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \mid \text{for each } \pi \in \beta, \text{ for each } \iota \in I_\Sigma, \text{ for each } \delta \in \theta, \right\} \text{ for } \langle \beta, \varrho, \lambda, \xi \rangle / \pi = \Delta^\Sigma_2(\delta).\]

Let \(\Sigma\) be a signature. We call \(M_\Sigma\) the morph admission function in \(\Sigma\).

**44 Definition.** For each signature \(\Sigma\),

\(\varrho\) is a canonical object in \(\Sigma\)

iff \(\varrho\) is a quintuple \(\langle \beta, \varrho, \lambda, \xi, \eta \rangle\),

\(\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma\), and

\(\eta \in \text{Quo}(\varrho)\).

Suppose that \(\Sigma\) is a signature. We write \(C_\Sigma\) for the set of canonical objects in \(\Sigma\). Suppose further that \(\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma\) and \(\pi \in \beta\). We write \(|\pi|_\varrho\) for the equivalence class of \(\pi\) in \(\varrho\). Thus, we write \(\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle\) for the canonical object \(\langle \beta, \varrho, \lambda, \xi, \eta \rangle\), where \(\eta\) is the equivalence class of \(\pi\) in \(\varrho\).

**45 Definition.** For each signature \(\Sigma\), for each \(\theta \in \mathcal{D}^\Sigma_0\), for each \(\langle \langle \beta, \varrho, \lambda, \xi, |\pi_1|_\varrho \rangle, \ldots, \rangle \langle \beta, \varrho, \lambda, \xi, |\pi_n|_\varrho \rangle \rangle \in (\mathcal{U}_\Sigma^\varrho)^*\),
\[ \langle \beta, \varrho, \lambda, \xi, |(\pi_1, \ldots, \pi_n)\rangle \text{ is an abbreviatory notation for} \]
\[ \langle \langle \beta, \varrho, \lambda, \xi, |\pi_1\rangle, \ldots, \langle \beta, \varrho, \lambda, \xi, |\pi_n\rangle \rangle. \]

46 Definition. For each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \), for each \( \theta \subseteq D_0^\Sigma \),
\[
U^\theta_\Sigma = \left\{ \langle \beta, \varrho, \lambda, \xi, \eta \rangle \in C_\Sigma \mid \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta) \right\},
\]
\[
S^\theta_\Sigma = \left\{ (u, \sigma) \in U^\theta_\Sigma \times S \mid \begin{array}{l}
\text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\
\text{for some } \pi \in \beta,
\end{array}
\begin{array}{l}
\quad u = \langle \beta, \varrho, \lambda, \xi, |\pi_\varrho\rangle,
\quad \sigma = \lambda(\pi),
\end{array}
\right\},
\]
\( A^\theta_\Sigma \) is the total function from \( A \) to \( \text{Pow}(U^\theta_\Sigma \times U^\theta_\Sigma) \) such that for each \( \alpha \in A, \)
\[
A^\theta_\Sigma(\alpha) = \left\{ (u, u') \in U^\theta_\Sigma \times U^\theta_\Sigma \mid \begin{array}{l}
\text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\
\text{for some } \pi \in \beta,
\end{array}
\begin{array}{l}
\quad u = \langle \beta, \varrho, \lambda, \xi, |\pi_\varrho\rangle \text{ and } \\
\quad u' = \langle \beta, \varrho, \lambda, \xi, |\pi_\alpha_\varrho\rangle
\end{array}\right\},
\]
\( R^\theta_\Sigma \) is the total function from \( R \) to the powerset of \( U^\theta_\Sigma \) such that for each \( \rho \in R, \)
\[
R^\theta_\Sigma(\rho) = \left\{ (u_1, \ldots, u_n) \in U^\theta_\Sigma \mid \begin{array}{l}
\text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\
\text{for some } \langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi,
\end{array}
\begin{array}{l}
\quad u_1 = \langle \beta, \varrho, \lambda, \xi, |\pi_1\rangle, \\
\quad \ldots, \\
\quad u_n = \langle \beta, \varrho, \lambda, \xi, |\pi_n\rangle
\end{array}\right\},
\]
\( I^\theta_\Sigma = \langle U^\theta_\Sigma, S^\theta_\Sigma, A^\theta_\Sigma, R^\theta_\Sigma \rangle. \)

47 Proposition. For each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \), for each \( \theta \subseteq D_0^\Sigma \),
\( U^\theta_\Sigma \) is a set,
\( S^\theta_\Sigma \) is a total function from \( U^\theta_\Sigma \) to \( S \),
\( A^\theta_\Sigma \) is a total function from \( A \) to the set of partial functions from \( U^\theta_\Sigma \) to \( U^\theta_\Sigma \),
\( R^\theta_\Sigma \) is a total function from \( R \) to the powerset of \( U^\theta_\Sigma \), and
\( I^\theta_\Sigma \) is a \( \Sigma \) interpretation.

Let \( \Sigma \) be a signature and \( \theta \subseteq D_0^\Sigma \), We call \( I^\theta_\Sigma \) the canonical \( \Sigma \) interpretation of \( \theta \).

48 Proposition. For each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \), for each \( \langle \beta, \varrho, \lambda, \xi, |\pi_\varrho\rangle \in C_\Sigma, \) for each total function \( \nu \) from \( \text{Var} \) to \( \beta/\pi \), for each \( \tau \in T^\Sigma, \)
\[ \text{if } \Pi^\Sigma(\tau) \text{ is defined and } \Pi^\Sigma(\tau) \in A^{**} \]
then \( \pi \Pi^\Sigma(\tau) \in \beta^{*}. \)

Proof. By arithmetic induction on the length of \( \tau. \)

49 Proposition. For each signature \( \Sigma \), for each \( \theta \subseteq D_0^\Sigma \),
$I^\theta_\Sigma$ is a $\langle \Sigma, \theta \rangle$ model.

**Proof.** Throughout this proof, suppose that, for each signature $\Sigma$, for each $\theta \subseteq D^\Sigma_0$, for each $o = \langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle \in U^\theta_\Sigma$, for each total function $\iota$ from $\mathbf{Var}$ to $\beta/\pi$,

$$\text{ass}_o^\iota$$ is the total function from $\mathbf{Var}$ to $U^\theta_\Sigma$ such that

for each $v \in \mathbf{Var}$, $\text{ass}_o^\iota(v) = \langle \beta, g, \lambda, \xi, |\pi \iota(v)|_\theta \rangle$.

Firstly, for each signature $\Sigma$, for each $\theta \subseteq D^\Sigma_0$, for each $o = \langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle \in U^\theta_\Sigma$, for each $\tau \in T^\Sigma$, for each total function $\iota$ from $\mathbf{Var}$ to $\beta/\pi$,

$$T_{I^\Sigma_0}^{\text{ass}}(\tau)(\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle)$$ is defined iff $\Pi^\Sigma_1(\tau)$ is defined and $\pi \Pi^\Sigma_2(\tau) \in \beta$, and

if $T_{I^\Sigma_0}^{\text{ass}}(\tau)(\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle)$ is defined

then $T_{I^\Sigma_0}^{\text{ass}}(\tau)(\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle) = \langle \beta, g, \lambda, \xi, |\pi \Pi^\Sigma_2(\tau)|_\theta \rangle$.

by proposition 48 and arithmetic induction on the length of $\tau$

Secondly, for each signature $\Sigma = \langle G, \subseteq, S, A, F, R, A R \rangle$, for each $\theta \subseteq D^\Sigma_0$, for each $o = \langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle \in U^\theta_\Sigma$, for each total function $\iota$ from $\mathbf{Var}$ to $\beta/\pi$,

for each $\tau \in T^\Sigma$, for each $\sigma \in G$,

$$\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle \in D^\Sigma_{I^\Sigma_0}(\tau \sim \sigma)$$

$$\iff T_{I^\Sigma_0}^{\text{ass}}(\tau)(\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle)$$ is defined and $S^\Sigma_{I^\Sigma_0}(T_{I^\Sigma_0}^{\text{ass}}(\tau)(\langle \beta, g, \lambda, \xi, |\tau|_\theta \rangle)) \subseteq \sigma$

$$\iff \Pi^\Sigma_2(\tau)$$ is defined, $\pi \Pi^\Sigma_2(\tau) \in \beta$, and $S^\Sigma_{I^\Sigma_0}(\langle \beta, g, \lambda, \xi, |\pi \Pi^\Sigma_2(\tau)|_\theta \rangle) \subseteq \sigma$

$$\iff \Pi^\Sigma_2(\tau)$$ is defined, and $\lambda/\pi(\Pi^\Sigma_2(\tau)) \subseteq \sigma$

$$\iff \langle \beta, g, \lambda, \xi \rangle / \pi \in \Delta^\Sigma_2(\tau \sim \sigma)$$

for each $\tau_1 \in T^\Sigma$, for each $\tau_2 \in T^\Sigma$,

$$\langle \beta, g, \lambda, \xi, |\tau_1|_\theta \rangle \in D^\Sigma_{I^\Sigma_0}(\tau_1 \approx \tau_2)$$

$$\iff T_{I^\Sigma_0}^{\text{ass}}(\tau_1)(\langle \beta, g, \lambda, \xi, |\tau_1|_\theta \rangle)$$ is defined, $T_{I^\Sigma_0}^{\text{ass}}(\tau_2)(\langle \beta, g, \lambda, \xi, |\tau_2|_\theta \rangle)$ is defined, and

$$T_{I^\Sigma_0}^{\text{ass}}(\tau_1)(\langle \beta, g, \lambda, \xi, |\tau_1|_\theta \rangle) = T_{I^\Sigma_0}^{\text{ass}}(\tau_2)(\langle \beta, g, \lambda, \xi, |\tau_2|_\theta \rangle)$$

$$\iff \Pi^\Sigma_2(\tau_1)$$ is defined, $\Pi^\Sigma_2(\tau_1) \in \beta$, $\Pi^\Sigma_2(\tau_2) \in \beta$, and

$$\langle \beta, g, \lambda, \xi, |\pi \Pi^\Sigma_2(\tau_1)|_\theta \rangle = \langle \beta, g, \lambda, \xi, |\pi \Pi^\Sigma_2(\tau_2)|_\theta \rangle$$

$$\iff \Pi^\Sigma_2(\tau_1)$$ is defined, $\Pi^\Sigma_2(\tau_2)$ is defined, and $\langle \Pi^\Sigma_2(\tau_1), \Pi^\Sigma_2(\tau_2) \rangle \in \overline{\rho/\pi}$

$$\iff \langle \beta, g, \lambda, \xi \rangle / \pi \in \Delta^\Sigma_2(\tau_1 \approx \tau_2)$$

for each $\rho \in R$, for each $x_1 \in \mathbf{Var}$, ..., for each $x_{AR(\rho)} \in \mathbf{Var}$,
\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in D^{\text{ass}}_{I^\Sigma}(\rho(x_1, \ldots, x_{AR(\rho)})) \]
\[ \iff \langle \text{ass}^\rho_\varrho(x_1), \ldots, \text{ass}^\rho_{AR(\rho)}(x_{AR(\rho)}) \rangle \in R^\rho_\Sigma(\rho) \]
\[ \iff \langle \langle \beta, \varrho, \lambda, \xi, \mid \pi_1(x_1) \mid \varrho \rangle, \ldots, \langle \beta, \varrho, \lambda, \xi, \mid \pi_{AR(\rho)}(x_{AR(\rho)}) \mid \varrho \rangle \rangle \in R^\varrho_\Sigma(\rho) \]
\[ \iff \langle \rho, \pi_1(x_1), \ldots, \pi_{AR(\rho)}(x_{AR(\rho)}) \rangle \in \xi \]
\[ \iff \langle \rho, \iota(x_1), \ldots, \iota(x_{AR(\rho)}) \rangle \in \xi / \pi \]
\[ \iff \langle \beta, \varrho, \lambda, \xi / \pi \rangle \in \Delta^\varrho_\Sigma(\rho(x_1, \ldots, x_{AR(\rho)})) \].

Thus, for each signature \( \Sigma \), for each \( \theta \subseteq D^\Sigma_0 \), for each \( \Theta = \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in U^\theta_\Sigma \), for each \( \delta \in D^\Sigma_0 \), for each total function \( \iota \) from \( \text{Var} \) to \( \beta / \pi \),

\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in D^{\text{ass}}_{I^\Sigma}(\delta) \iff \langle \beta, \varrho, \lambda, \xi / \pi \rangle \in \Delta^\varrho_\Sigma(\delta) \].

by arithmetic induction on the complexity of \( \delta \)

(since, for each \( \pi' \in \beta / \pi \), \( \text{ass}^\rho_{\varrho, \lambda, \xi, \mid \pi' \mid \varrho} \rho = \text{ass}^\rho_{\varrho, \lambda, \xi, \mid \pi' \mid \varrho} \)

Thus, for each signature \( \Sigma \), for each \( \theta \subseteq D^\Sigma_0 \), for each \( \Theta = \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in C^\Sigma_\Sigma \),

\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in U^\theta_\Sigma, \]
\[ \implies \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in U^\theta_\Sigma, \langle \beta, \varrho, \lambda, \xi \rangle \in M^\theta_\theta(\theta) \) and \( \pi \in \beta \)
\[ \implies \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in U^\theta_\Sigma, \]
\[ \text{for each } \delta \in \theta, \text{ for each total function } \iota \text{ from } \text{Var} \text{ to } \beta / \pi, \]
\[ \langle \beta, \varrho, \lambda, \xi / \pi \rangle \in \Delta^\varrho_\Sigma(\delta) \]
\[ \implies \text{for each } \delta \in \theta, \text{ for each total function } \iota \text{ from } \text{Var} \text{ to } \beta / \pi, \]
\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in D^{\text{ass}}_{I^\Sigma}(\delta) \]
\[ \implies \text{for each } \delta \in \theta, \text{ for some } \text{ass} \in \text{Ass}_{I^\Sigma}^\theta, \]
\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in D^{\text{ass}}_{I^\Sigma}(\delta) \]
\[ \implies \text{for each } \delta \in \theta, \text{ for each } \text{ass} \in \text{Ass}_{I^\Sigma}^\theta, \]
\[ \langle \beta, \varrho, \lambda, \xi, \mid \pi \mid \varrho \rangle \in \Delta^\varrho_\Sigma(\delta) \]
\[ \implies \langle \beta, \varrho, \lambda, \xi / \pi \rangle \in \Theta^\Sigma_{I^\Sigma}(\theta). \]

\[ \blacksquare \]

50 Definition. For each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle \), for each \( \Sigma \) interpretation \( I = \langle U, S, A, R \rangle \),

\( A_I \) is the binary relation on \( U \times M_\Sigma \) such that, for each \( u \in U \), for each \( \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma \), \( \langle u, \langle \beta, \varrho, \lambda, \xi \rangle \rangle \in A_I \) if

\[ \ldots \]
3 EXISTENCE PROOF FOR EXHAUSTIVE MODELS

\[ \beta = \{ \pi \in A^* \mid T_I(\pi, u) \text{ is defined} \}, \]

\[ \rho = \{ (\pi_1, \pi_2) \in A^* \times A^* \mid \begin{array}{l}
T_I(\pi_1, u) \text{ is defined,} \\
T_I(\pi_2, u) \text{ is defined, and} \\
T_I(\pi_1, u) = T_I(\pi_2, u) \end{array} \}, \]

\[ \lambda = \{ (\pi, \sigma) \in A^* \times S \mid \begin{array}{l}
T_I(\pi, u) \text{ is defined,} \\
S(T_I(\pi, u)) = \sigma \end{array} \}, \]

\[ \xi = \{ (\rho, \pi_1, \ldots, \pi_n) \in R \times (A^*)^n \mid \begin{array}{l}
T_I(\pi_1, u) \text{ is defined,} \\
T_I(\pi_n, u) \text{ is defined, and} \\
\langle T_I(\pi_1, u), \ldots, T_I(\pi_n, u) \rangle \in R(\rho) \end{array} \}. \]

51 Proposition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I = \langle U, S, A, R \rangle \),

\( A_I \) is a total function from \( U \) to \( \mathcal{M}_\Sigma \).

52 Proposition. For each signature \( \Sigma \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \),

for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( o_1 \in U_1 \), for each \( o_2 \in U_2 \),

\[ A_{I_1}(o_1) = A_{I_2}(o_2) \]

iff \( (o_1, I_1) \) and \( (o_2, I_2) \) are congruent in \( \Sigma \).

Proof. Firstly, for each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, A R \rangle \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( o_1 \in U_1 \), for each \( o_2 \in U_2 \),

\[ A_{I_1}(o_1) = A_{I_2}(o_2) \]

\[ \implies \{ (o'_1, o'_2) \in \overline{U}_1 \times \overline{U}_2 \mid \begin{array}{l}
\text{for some } \pi \in \overline{A}^*, \\
T_{I_1}(\pi, o_1) \text{ is defined,} \\
T_{I_2}(\pi, o_2) \text{ is defined,} \\
o'_1 = T_{I_1}(\pi, o_1), \text{ and} \\
o'_2 = T_{I_2}(\pi, o_2) \end{array} \} \]

is a congruence from \( (o_1, I_1) \) to \( (o_2, I_2) \) in \( \Sigma \)

\[ \implies (o_1, I_1) \text{ and } (o_2, I_2) \text{ are congruent in } \Sigma. \]

Secondly, for each signature \( \Sigma = \langle G, \subseteq, S, A, F, R, A R \rangle \), for each \( \Sigma \) interpretation \( I_1 = \langle U_1, S_1, A_1, R_1 \rangle \), for each \( \Sigma \) interpretation \( I_2 = \langle U_2, S_2, A_2, R_2 \rangle \), for each \( o_1 \in U_1 \), for each \( o_2 \in U_2 \),

\( (o_1, I_1) \) and \( (o_2, I_2) \) are congruent in \( \Sigma \)

\[ \implies \text{for some } f, f \text{ is a congruence from } (o_1, I_1) \text{ to } (o_2, I_2) \]
for some congruence $f$ from $\langle o_1, I_1 \rangle$ to $\langle o_2, I_2 \rangle$, for each $\pi \in A^*$,

$T_{I_1}(\pi, o_1)$ is defined iff $T_{I_2}(\pi, o_2)$ is defined, and

if $T_{I_1}(\pi, o_1)$ is defined then $f(T_{I_1}(\pi, o_1)) = T_{I_2}(\pi, o_2)$

by induction on the length of $\pi$

$\implies A_{I_1}(o_1) = A_{I_2}(o_2)$.

53 Lemma. For each signature $\Sigma$, for each $\langle \beta, \rho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\tau \in T^\Sigma$, for each $\iota_1 \in I_\Sigma$, for each $\iota_2 \in I_\Sigma$,

if for each $v \in \text{Var}$, $\iota_1(v) \in \overline{\beta}$ and $\iota_2(v) \in \overline{\beta}$, and

for each $v \in \text{FV}(\tau)$, $\langle \iota_1(v), \iota_2(v) \rangle \in \hat{\delta}$

then $\Pi^\iota_1(\tau)$ is defined and $\Pi^\iota_2(\tau)$ is defined and $\Pi^\iota_2(\tau) \in \overline{\beta}$, and

if $\Pi^\iota_1(\tau)$ is defined and $\Pi^\iota_2(\tau) \in \overline{\beta}$ then $\langle \Pi^\iota_1(\tau), \Pi^\iota_2(\tau) \rangle \in \hat{\delta}$, and

for each $\delta \in D^\Sigma$, for each $\iota_1 \in I_\Sigma$, for each $\iota_2 \in I_\Sigma$,

if for each $v \in \text{Var}$, $\iota_1(v) \in \overline{\beta}$ and $\iota_2(v) \in \overline{\beta}$, and

for each $v \in \text{FV}(\delta)$, $\langle \iota_1(v), \iota_2(v) \rangle \in \hat{\delta}$

then $\langle \beta, \rho, \lambda, \xi \rangle \in \Delta^\iota_1(\delta)$ iff $\langle \beta, \rho, \lambda, \xi \rangle \in \Delta^\iota_2(\delta)$.

Proof. By proposition 48 and induction on the length of $\tau$, and by induction on the complexity of $\delta$, respectively.

54 Definition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\iota \in I_\Sigma$,

$\text{ass}^\iota_{o,I} = \left\{ \langle v, o' \rangle \in \text{Var} \times U \mid T_I(\Pi_\Sigma(v), o) \text{ is defined, and } T_I(\Pi_\Sigma(v), o) = o' \right\}$.

55 Proposition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\langle \beta, \rho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\iota \in I_\Sigma$,

if for each $v \in \text{Var}$, $\iota(v) \in \overline{\beta}$, and

$A_I(o) = \langle \beta, \rho, \lambda, \xi \rangle$

then $\text{ass}^\iota_{o,I} \in \text{Ass}_I$.

56 Lemma. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\langle \beta, \rho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\tau \in T^\Sigma$, for each $\iota \in I_\Sigma$,
if for each $v \in \text{Var}$, $\iota(v) \in \overline{\beta}$, and

$$A_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$$

then $T^\text{ass}_{o,I}(\tau)(o)$ is defined iff $T_I(\Pi^\Sigma_\Delta(\tau), o)$ is defined, and

if $T^\text{ass}_{o,I}(\tau)(o)$ is defined then $T^\text{ass}_{o,I}(\tau)(o) = T_I(\Pi^\Sigma_\Delta(\tau), o)$.

**Proof.** By proposition 48 and induction on the length of $\tau$. ■

57 **Proposition.** For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\delta \in \mathcal{D}^\Sigma$, for each $\iota \in I_\Sigma$,

if for each $v \in \text{Var}$, $\iota(v) \in \overline{\beta}$, and

$$A_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$$

then $o \in D^\text{ass}_{o,I}(\delta)$ iff $\langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\Sigma_\Delta(\delta)$.

**Proof.** For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, for each $\iota \in I_\Sigma$,

if for each $v \in \text{Var}$, $\iota(v) \in \overline{\beta}$, and

$$A_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$$

then for each $o' \in \overline{\text{Co}}^\Sigma_I$, for each $v \in \text{Var}$, $\text{ass}_{o,I}^\Sigma(v) = \text{ass}_{o,I}^\Sigma(\overline{\#(o')})$, and

for each $\pi \in \overline{\beta}$, for each $o' \in \overline{\text{Co}}^\Sigma_I$,

$$o' = T_I(\pi, o)$$

$$\implies T_I(\overline{\#(o')}, o) = T_I(\overline{\pi}, o)$$

$$\implies (\overline{\#(o')}, \overline{\pi}) \in \overline{\varrho}$$

$$\implies \text{for each } \delta \in \mathcal{D}^\Sigma,$$

$$\langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\Sigma_\Delta(\overline{\#(o')}) \text{ iff } \langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\Sigma_\Delta(\overline{\delta}).$$

by lemma 53

Using this result, the proposition follows by lemma 56 and induction on the complexity of $\delta$. ■

58 **Lemma.** For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle$, for each interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\pi \in A^*$, for each $\pi' \in A^*$,

$T_I(\pi \pi', o)$ is defined iff $T_I(\pi', T_I(\pi, o))$ is defined, and

if $T_I(\pi \pi', o)$ is defined then $T_I(\pi \pi', o) = T_I(\pi', T_I(\pi, o))$. 
3. EXISTENCE PROOF FOR EXHAUSTIVE MODELS

Proof. By induction on the length of $\pi'$.

59 Lemma. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle$, for each interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\pi \in A^*$, for each $\pi' \in A^*$,

$T_I(\pi'\pi', o)$ is defined iff $T_I(\pi', T_I(\pi, o))$ is defined, and

if $T_I(\pi'\pi', o)$ is defined then $T_I(\pi'\pi', o) = T_I(\pi', T_I(\pi, o))$.

Proof. Follows from lemma 58.

60 Proposition. For each signature $\Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\pi \in A^*$,

if $T_I(\pi, o)$ is defined then $A_I(T_I(\pi, o)) = A_I(o)/\pi$.

Proof. Uses lemma 58 and lemma 59.

61 Proposition. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,

if $I$ is a $\langle \Sigma, \theta \rangle$ model
then for each $o \in U$, $A_I(o) \in M_\Sigma(\theta)$.

Proof. For each signature $\Sigma$, for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,

$I$ is a $\langle \Sigma, \theta \rangle$ model

$\implies$ for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma$,

$\langle \beta, \varrho, \lambda, \xi \rangle = A_I(o)$

$\implies$ for each $\pi \in \beta$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = A_I(T_I(\pi, o))$ by proposition 60

$\implies$ for each $\pi \in \beta$, for each ass $\in \text{Ass}_I$, for each $\delta \in \theta$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = A_I(T_I(\pi, o))$ and $T_I(\pi, o) \in D_I^{\text{ass}}(\delta)$

$\implies$ for each $\pi \in \beta$, for each total function $\nu: \text{Var} \rightarrow \beta/\pi$, for each $\delta \in \theta$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = A_I(T_I(\pi, o))$ and $T_I(\pi, o) \in D_I^{\text{ass}}(\nu(T_I(\pi, o)), \delta)$ by proposition 55

$\implies$ for each $\pi \in \beta$, for each total function $\nu: \text{Var} \rightarrow \beta/\pi$, for each $\delta \in \theta$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta_\Sigma(\delta)$ by proposition 57

$\implies$ for each $\pi \in \beta$, for some $\nu \in \text{I}_\Sigma$, for each $\delta \in \theta$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta_\Sigma(\delta)$ since $\beta/\pi \neq \emptyset$
for each $\pi \in \beta$, for each $i \in I_\Sigma$, for each $\delta \in \theta$, by proposition 42

$$\langle \beta, \rho, \lambda, \xi \rangle/\pi \in \Delta_\Sigma^i(\delta)$$

$$\Rightarrow \langle \beta, \rho, \lambda, \xi \rangle \in M_\Sigma(\theta)$$

for each $o \in U$, $A_I(o) \in M_\Sigma(\theta)$. by proposition 51

62 Proposition. For each signature $\Sigma$, for each $\theta \subseteq D_0^\Sigma$, for each $\langle \beta, \rho, \lambda, \xi \rangle \in M_\Sigma(\theta)$,

$$A_{I_\Sigma}^g(\langle \beta, \rho, \lambda, \xi, |e|_\theta \rangle) = \langle \beta, \rho, \lambda, \xi \rangle.$$  

Proof. Follows from the observation that, for each signature $\Sigma = \langle G, \subseteq, S, A, F, R, AR \rangle$, for each $\theta \subseteq D_0^\Sigma$, for each $\langle \beta, \rho, \lambda, \xi, |e|_\theta \rangle \in U_\Sigma^g$, for each $\pi \in A^*$,

$\pi \in \beta$ iff $T_{I_\Sigma}(\pi, \langle \beta, \rho, \lambda, \xi, |e|_\theta \rangle)$ is defined, and

if $\pi \in \beta$ then $T_{I_\Sigma}(\pi, \langle \beta, \rho, \lambda, \xi, |e|_\theta \rangle) = \langle \beta, \rho, \lambda, \xi, |\pi|_\theta \rangle$. by induction on the length of $\pi$

63 Proposition. For each signature $\Sigma$, for each $\theta \subseteq D_0^\Sigma$,

$I_\Sigma^g$ is an exhaustive $(\Sigma, \theta)$ model.

Proof. For each signature $\Sigma$, for each $\theta \subseteq D_0^\Sigma$, $I_\Sigma^g$ is a $(\Sigma, \theta)$ model, and for each $\Sigma$ interpretation $I = \langle U, S, A, R \rangle$,

$I$ is a $(\Sigma, \theta)$ model

$$\Rightarrow$$ for each $o \in U$, $A_I(o) \in M_\Sigma(\theta)$ by proposition 61

$$\Rightarrow$$ for each $o \in U$, for some $\langle \beta, \rho, \lambda, \xi \rangle \in M_\Sigma(\theta)$,

$A_I(o) = \langle \beta, \rho, \lambda, \xi \rangle$, and

$A_{I_\Sigma}^g(\langle \beta, \rho, \lambda, \xi, |e|_\theta \rangle) = \langle \beta, \rho, \lambda, \xi \rangle$ by proposition 62

$$\Rightarrow$$ for each $o \in U$, for some $o' \in U_\Sigma^g$,

$\langle o', I_\Sigma^g \rangle$ and $\langle o, I \rangle$ are congruent in $\Sigma$ by proposition 52

$$\Rightarrow I_\Sigma^g$ simulates $I$ in $\Sigma$

$$\Rightarrow I_\Sigma^g$ is an exhaustive $(\Sigma, \theta)$ model. by theorem 32

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64 Theorem. For each signature $\Sigma$, for each theory $\theta$, there exists a $\Sigma$ interpretation $I$ such that

$I$ is an exhaustive $(\Sigma, \theta)$ model.

An immediate corollary of Theorem 64 and the definition of an exhaustive model 22 is that if a grammar has a non-empty model then it has a non-empty exhaustive model.
References


