Mathematics for linguists

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Inferences and truth trees

- Inferences (with a finite set of premises; from now on we tacitly assume that premise sets are finite) can always be tranformed into tautologies using the deduction theorem
- Inferences can also directly be proved using truth trees though:
  - premises are assumed to be true
  - conclusion is assumed to be false

Inferences and truth trees

- to prove the inference

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi,$$

start your truth tree with

$$\begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_n \\ \neg\psi \end{array}$$

Inferences and truth trees

Theorem 6 *Let* $\varphi_1, \ldots, \varphi_n$ *be formulas of statement logic.* ψ *follows logically from the premises* $\varphi_1, \ldots, \varphi_n$ *if every branch of a truth tree which starts with* $\varphi_1, \ldots, \varphi_n$ *and* ψ *and only uses the known rules, can be closed with an "x" because every formula occurs in it both in negated and non-negated form.*

Example

$$p \rightarrow q, \neg q \Rightarrow \neg p$$

1.	$p \rightarrow q$	(A)
2.	$\neg q$	(A)
3.	$\neg\neg p$	(A)

4.	$\neg p$	(1)	5.	q	(1)
	x	$(3, 4)$		x	$(2, 5)$

Example

- Inference

$$p \rightarrow q, p \vee r, \neg r \Rightarrow p \wedge q$$

- there is more than one way to prove this

Example

1.	$p \longrightarrow q$	(A)
2.	$p \vee r$	(A)
3.	$\neg r$	(A)
4.	$\neg(p \wedge q)$	(A)

- 5. p (2)
  - 7. $\neg p$ (1)
    x $(5, 7)$
  - 8. q (1)
    - 9. $\neg p$ (4)
      x $(5, 9)$
    - 10. $\neg q$ (4)
      x $(8, 10)$
- 6. r (2)
  x $(2, 6)$

Example

1.	$p \rightarrow q$	(A)
2.	$p \vee r$	(A)
3.	$\neg r$	(A)
4.	$\neg(p \wedge q)$	(A)

- 5. $\neg p$ (1)
  - 7. p (2)
    × $(5, 7)$
  - 8. r (2)
    × $(3, 8)$
- 6. q (1)
  - 9. p (2)
    - 11. $\neg p$ (4)
      × $(9, 11)$
    - 12. $\neg q$ (4)
      × $(6, 12)$
  - 10. r (2)
    × $(3, 10)$

Natural deduction: motivation

- proving theorems via truth trees is sometimes tedious
- intuitive content of the operators of statement logic is not directly transparent
- for instance, some inferences are obvious from this intutive content:

$$
\begin{array}{rcl}
\varphi, \psi & \Rightarrow & \varphi \wedge \psi \\
\varphi \wedge \psi & \Rightarrow & \varphi \\
\varphi, \varphi \rightarrow \psi & \Rightarrow & \psi \\
\varphi \rightarrow \psi, \psi \rightarrow \varphi & \Rightarrow & \varphi \leftrightarrow \psi \\
& \vdots &
\end{array}
$$

Natural deduction: motivation

- meta-logical properties of the inference relation cannot be used
  - identity:

$$\varphi \Rightarrow \varphi$$

  - cut:

$$\frac{M \Rightarrow \varphi \qquad N, \varphi \Rightarrow \xi}{M, N \Rightarrow \xi}$$

  - monotonicity:

$$\frac{M \Rightarrow \varphi}{M, \psi \Rightarrow \varphi}$$

Natural deduction: motivation

- **Calculus of natural deduction:**
  - *syntactic* calculus: only the syntactic form of the formula matters (so the calculus of truth trees is also syntactic, despite its name)
  - two central issues for each operator O:
    - When is is possible to use O in the conclusion of an inference? (introduction rule)
    - What can I do with a premise that contains O as main functor? (elimination rule)

Natural deduction: motivation

- Examples for introduction rules:

$$\frac{M \Rightarrow \varphi \qquad M \Rightarrow \psi}{M \Rightarrow \varphi \wedge \psi}$$

$$\frac{M, \varphi \Rightarrow \psi}{M \Rightarrow \varphi \rightarrow \psi}$$

- Examples for elimination rules

$$\frac{M \Rightarrow \varphi \wedge \psi}{M \Rightarrow \varphi}$$

$$\frac{M \Rightarrow \varphi \rightarrow \psi \qquad M \Rightarrow \varphi}{M \Rightarrow \psi}$$

Calculus of natural deduction

- Notation: we use $\vdash$ (rather than $\Rightarrow$) for syntactically derived inferences
- Terminology:
  - syntactically proven formulas are called **theorems** (which is the *counterpart to the semantic notion of a* ***tautology***)
  - If the conclusion φ can be syntactically derived from the premises M, then φ is **derivable** from M *(counterpart to the semantic notion "**follows logically**")*

Natural deduction

- basic structure of a proof (in the calculus of natural deduction):

premises
intermediate steps ⋮ intermediate steps conclusion

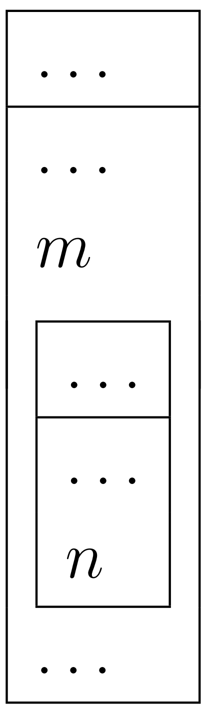
Natural deduction

- **intermediate steps** are
  - formulas that can be derived from preceding lines (within the same box or within including boxes) by applying an introduction rule or an elimination rule, or
  - complete proofs (i.e. boxes)
  - copies of preceding lines

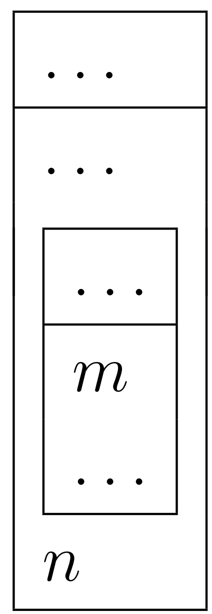
Accessibility

- Every line in a proof is included by a set of boxes.
- Relative to a certain line n, another line m is **accessible** if
  - m precedes n, and
  - all boxes that include m also include n

good:



bad:



Natural deduction

- Rules: for every operator of statement logic, there are one or two **introduction** rules and one or two **elimination** rules
- Notation:
  - at least one formula or box above the horizontal line
  - one formula below the horizontal line
  - name of the rule is written next to the line

Natural deduction

- Rule application: if all formulas/boxes over the line occur in a proof and are **accessible**, then the formula below the line may be added to the proof
- formulas in a proof are numbered
- the numbers of the used premises are written behind the new formula

Natural deduction: rules

Negation

$$\dfrac{\boxed{\begin{array}{l} \varphi \\ \hline \vdots \\ \psi \\ \neg\psi \end{array}}}{\neg\varphi}\ \neg I$$

$$\dfrac{\neg\neg\varphi}{\varphi}\ \neg E$$

Natural deduction: rules

Conjunction

$$\frac{\begin{array}{c}\varphi \\ \psi\end{array}}{\varphi \wedge \psi} \wedge I$$

$$\frac{\varphi \wedge \psi}{\varphi} \wedge E1 \qquad \frac{\varphi \wedge \psi}{\psi} \wedge E2$$

Natural deduction: rules

Disjunction

$$\frac{\varphi}{\varphi \vee \psi}\,\vee I1 \qquad\qquad \frac{\varphi}{\psi \vee \varphi}\,\vee I2$$

$$\frac{\varphi \vee \psi \qquad \begin{array}{|c|}\hline \varphi \\ \hline \vdots \\ \xi \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \psi \\ \hline \vdots \\ \xi \\ \hline \end{array}}{\xi}\,\vee E$$

Natural deduction: rules

Implication

$$\frac{\boxed{\begin{array}{c}\varphi \\ \hline \vdots \\ \psi\end{array}}}{\varphi \longrightarrow \psi} \longrightarrow I$$

$$\frac{\begin{array}{c}\varphi \longrightarrow \psi \\ \varphi\end{array}}{\psi} \longrightarrow E$$

Natural deduction: rules

Equivalence

$$\frac{\begin{array}{|c|}\hline \varphi \\ \hline \vdots \\ \psi \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \psi \\ \hline \vdots \\ \varphi \\ \hline\end{array}}{\varphi \leftrightarrow \psi} \leftrightarrow I$$

$$\frac{\begin{array}{c}\varphi \leftrightarrow \psi \\ \varphi\end{array}}{\psi} \leftrightarrow E, 1 \qquad\qquad \frac{\begin{array}{c}\varphi \leftrightarrow \psi \\ \psi\end{array}}{\varphi} \leftrightarrow E, 2$$

Natural deduction

Definition 7 *If it is possible to construct a proof of the form*

$$\begin{array}{|c|} \varphi_1 \\ \vdots \\ \varphi_n \\ \hline \vdots \\ \psi \\ \hline \end{array}$$

*according to the rules of natural deduction, then ψ is **derivable** from $\varphi_1, \ldots, \varphi_n$, i.e.*

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

Natural deduction

Theorem 8 (Soundness and completeness)

$$M \vdash \varphi$$

if and only if

$$M \Rightarrow \varphi$$

Examples: de Morgan's Laws (1)

1. $\neg(p \wedge q)$ $\quad (A)$
    2. $\neg(\neg p \vee \neg q)$ $\quad (A)$
        3. $\neg p$ $\quad (A)$
        4. $\neg p \vee \neg q$ $\quad \vee I1; 3$
    5. $\neg\neg p$ $\quad \neg I; 3, 4, 2$
        6. $\neg q$ $\quad (A)$
        7. $\neg p \vee \neg q$ $\quad \vee I2; 6$
    8. $\neg\neg q$ $\quad \neg I; 6, 7, 2$
    9. p $\quad \neg E; 5$
    10. q $\quad \neg E; 8$
    11. $p \wedge q$ $\quad \wedge I; 9, 10$
12. $\neg\neg(\neg p \vee \neg q)$ $\quad \neg I; 2, 11, 1$
13. $\neg p \vee \neg q$ $\quad \neg E; 12$

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

Examples: de Morgan's Laws (2)

$1. \neg p \vee \neg q \qquad (A)$

$2. p \wedge q \qquad (A)$

$3. p \qquad \wedge I 1; 2$

$4. q \qquad \wedge I 2; 2$

$5. \neg p \qquad (A)$

$6. \neg p \qquad (6)$

$7. \neg q \qquad (A)$

$8. p \qquad (A)$

$9. p \qquad 8$

$10. \neg p \qquad \neg I; 8, 4, 7$

$11. \neg p \qquad \vee E; 1, 5, 6, 7, 9$

$12. \neg(p \wedge q) \qquad \neg I; 2, 3, 11$

$$\neg p \vee \neg q \vdash \neg(p \wedge q)$$

Examples: de Morgan's Laws (3)

1. $\neg(p \vee q)$	(A)
2. p	(A)
3. $p \vee q$	$\vee I1; 2$
4. $\neg p$	$\neg I; 2, 1, 3$
5. q	(A)
6. $p \vee q$	$\vee I2; 5$
7. $\neg q$	$\neg I; 5, 1, 6$
8. $\neg p \wedge \neg q$	$\wedge I; 4, 7$

$$\neg(p \vee q) \vdash \neg p \wedge \neg q$$

Examples: de Morgan's Laws (4)

1. $\neg p \wedge \neg q$ (A)
2. $\neg p$ $\wedge I1; 1$
3. $\neg q$ $\wedge I2; 1$
4. $p \vee q$ (A)
5. p (A)
6. p 5
7. q (A)
8. $\neg p$ (A)
9. $\neg p$ 8
10. $\neg\neg p$ $\neg I; 8, 3, 7$
11. p $\neg E; 10$
12. p $\vee E; 4, 5, 6, 7, 11$
13. $\neg(p \vee q)$ $\neg I; 4, 2, 12$

$$\neg p \wedge \neg q \vdash \neg(p \vee q)$$

Lemmas

- Cut rule:

$$\frac{M \Rightarrow \varphi \qquad N, \varphi \Rightarrow \xi}{M, N \Rightarrow \xi}$$

- if a derivation has been proved once, it can be re-used
- massively simplifies work

Ex falsum quod libet

$1. \varphi$	(A)
$2. \neg\varphi$	(A)
$3. \neg\psi$	(A)
$4. \neg\neg\psi$	$\neg I; 3, 1, 2$
$5. \psi$	$\neg E; 4$

$$\varphi, \neg\varphi \vdash \psi$$

- this inference, once proved, can be used as a new rule
- if, at some stage in a proof, both and $\neg\varphi$ are accessible (for any formula φ), any other formula may be added