Mathematics for linguists

WS 2009/2010

University of Tübingen

February 4, 2010

Gerhard Jäger

Natural deduction for predicate logic

- direct extension of natural deduction for statement logic
- four new rules: one introduction rule and one elimination rule for each quantifier
- there are side conditions that need to be taken into account

Natural deduction: rules

Universal quantifier

$$\frac{\varphi}{\forall v \varphi} \forall E$$

- v is an arbitrary variable
- **Constraint:** v does not occur free in any accessible assumption!

$$\frac{\forall v \varphi}{[t/v]\varphi} \forall B$$

- v is an arbitrary variable and t an arbitrary constant or variable
- **Constraint:** if t is a variable, it must not occur bound in $[t/v]\varphi$

Natural deduction: rules

Existential quantifier

$$\frac{[t/v]\varphi}{\exists v \varphi}\ \exists E$$

- v is an arbitrary variable and t an arbitrary constant or variable
- **Constraint:** if t is a variable, it must not occur bound in $[t/v]\varphi$

Natural deduction: rules

Existential quantifier

$$\dfrac{\exists v\varphi \qquad \begin{array}{|c|}\hline [c/v]\varphi \\ \hline \vdots \\ \psi \\ \hline \end{array}}{\psi}\ \exists B$$

- v is an arbitrary variable
- Constraints
  - c is a new constant that does not occur so far in the proof
  - c does not occur in ψ

Examples

$\neg\exists x Px \vdash \forall x \neg Px$

> $1. \neg\exists x P(x) \quad (A)$
>
> > $2. Px \quad (A)$
> >
> > $3. \exists x Px \quad 2; \exists I$
>
> $4. \neg Px \quad 2, 3, 1, 3; \neg I$
>
> $5. \forall x \neg Px \quad 4; \forall I$

$\forall x \neg Px \vdash \neg\exists x Px$

> $1. \forall x Px \quad (A)$
>
> > $2. \exists x Px \quad (A)$
> >
> > > $3. Pa \quad (A)$
> > >
> > > > $4. \forall x \neg Px \quad (A)$
> > > >
> > > > $5. \neg Pa \quad 4; \forall E$
> > >
> > > $6. \neg\forall x \neg Px \quad 4, 5, 3, 5; \neg I$
> >
> > $7. \neg\forall x \neg Px \quad 2, 3, 4;$
>
> $8. \neg\exists x Px \quad 2, 1, 7; \neg I$

Examples

$\neg\forall x Px \vdash \exists x \neg Px$

1. $\neg\forall x Px$ — (A)
   - 2. $\neg\exists x \neg Px$ — (A)
     - 3. $\neg Px$ — (A)
     - 4. $\exists x \neg Px$ — $3; \exists I$
     - 5. $\neg\neg Px$ — $3, 4, 2; \neg I$
     - 6. Px — $\neg E$
     - 7. $\forall x Px$ — $6; \forall I$
   - 8. $\neg\neg\exists x \neg Px$ — $2, 7, 1; \neg I$
   - 9. $\exists x \neg Px$ — $\neg E$

$\exists x \neg Px \vdash \neg\forall x Px$

1. $\exists x \neg Px$ — (A)
   - 2. $\forall x Px$ — (A)
     - 3. $\neg Pa$ — (A)
       - 4. $\exists x \neg Px$ — (A)
       - 5. Pa — $2; \forall E$
     - 6. $\neg\exists x \neg Px$ — $4, 3, 5; \neg I$
     - 7. $\neg\exists x \neg Px$ — $1, 2, 3; \exists E$
   - 8. $\neg\forall x Px$ — $2, 3, 1; \neg I$

Examples

$\forall x(Px \wedge Qx) \vdash \forall x Px \wedge \forall x Qx$

$1. \forall x(Px \wedge Qx)$	(A)
$2. Px \wedge Qx$	$1; \forall E$
$3. Px$	$2; \wedge E1;$
$4. Qx$	$2; \wedge E2;$
$5. \forall x Px$	$3; \forall I$
$6. \forall x Qx$	$4; \forall I$
$7. \forall x Px \wedge \forall x Qx$	$5, 4, \wedge I$

$\exists x Px \rightarrow Qa \vdash \forall x(Px \rightarrow Qa)$

$1. \exists x Px \rightarrow Qa$	(A)
$2. Px$	(A)
$3. \exists x Px$	$2; \exists I$
$4. Qa$	$1, 2; \rightarrow E$
$5. Px \rightarrow Qa$	$2, 3; \rightarrow I$
$6. \forall x(Px \rightarrow Qa)$	$5; \forall I$

Final remarks

- calculus of natural deduction is sound and complete
- this means that all and only the logically valid inferences can be proved
- the constraints are necessary; otherwise it would be possible to derive invalid inferences, for instance
  - $\exists x Px \vdash \forall x Px$

Final remarks

- As for the truth tree method, there is no fool-proof solution strategy for natural deduction; and for the same reason
- with the elimination rule for the existential quantifier, arbitrarily many constants can be introduced into a proof, and each constant can be used in the elimination rule for the universal quantifier