Static and dynamic stability conditions for structurally stable signaling games

Gerhard Jäger
Gerhard.Jaeger@uni-bielefeld.de

September 8, 2007

Workshop on Communication, Game Theory, and Language, NWU
Overview

- signaling games
- costly signaling
- some examples
- conditions for evolutionary stability
- ESSets
- neutral stability
- dynamic stability and basins of attraction
Signaling games

**general setup**

- two players, the sender and the receiver.
- sender has private information about an event that is unknown to the receiver
- event is chosen by nature according to a certain fixed probability distribution
- sender emits a signal which is revealed to the receiver
- receiver performs an action, and the choice of action may depend on the observed signal
- utilities of sender and receiver may depend on the event, the signal and the receiver’s action
Signaling games

**specific assumptions**

- the utility of sender and receiver are identical,
- set of events $\mathcal{E}$, set of events $\mathcal{F}$, and set of actions $\mathcal{A}$ are finite,
- $\mathcal{E} = \mathcal{A}$ (the receiver’s action is to guess an event)
Signaling games

<table>
<thead>
<tr>
<th>costly signaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>▪ production/reception of signals may incur costs</td>
</tr>
<tr>
<td>▪ examples:</td>
</tr>
<tr>
<td>▪ length, processing complexity etc. of natural language expressions</td>
</tr>
<tr>
<td>▪ advertising costs in economics</td>
</tr>
<tr>
<td>▪ “handicap” signaling in biology</td>
</tr>
<tr>
<td>▪ ...</td>
</tr>
<tr>
<td>▪ can be represented as negative utility</td>
</tr>
</tbody>
</table>
Signaling games

- let $e$ be the event to be communicated, $\sigma$ the signal and $a$ the receiver’s action
- $c_\sigma$ is the cost of using signal $\sigma$
- partnership game: $S$ and $H$ have identical utility function

utility function (extensive form)

$$u(e, \sigma, a) = \delta_{e,a} + c_\sigma$$ (1)
Signaling games

**Matrix representation**

- Let $n = |\mathcal{E}|$ be the number of events.
- $m = |\mathcal{F}|$ is the number of signals.
- (Pure) strategies can be represented as matrices with one 1 per row and else columns.
- Sender strategy $S$: $n \times m$-matrix.
- Receiver strategy $R$: $m \times n$-matrix.
- $\vec{e}$: nature’s probability distribution over events.
- $\vec{c}$: costs of signals $1, \ldots, m$. 
Signaling games

normal form utility function

\[ u(S, R) = \sum_i e_i \times \sum_j s_{ij} (r_{ji} + c_j) \]  (2)
Signaling games

compiling costs and probabilities into matrix notation

\[ p_{ij}^S \doteq s_{ij} \times e_i \]
\[ q_{ij}^R \doteq r_{ij} + c_i \]

utility function

\[ u(S, R) = \sum_i \sum_j p_{ij}^S q_{ji}^R = \text{tr}(P^S Q^R). \]
let $x$ be a mixed strategy of a symmetrized signaling game with costly signaling

\[
P^x = \sum_{P,Q} x(P, Q)P \quad (3)
\]

\[
Q^x = \sum_{P,Q} x(P, Q)Q \quad (4)
\]
Signaling games

symmetrized utility function

\[ u(x, y) = \text{tr}(P^x Q^y) + \text{tr}(P^y Q^x) \]  \hspace{1cm} (5)
Signaling games

**Further constraints**

- Costs are normalized such that $\max_i c_i = 0$
- All events have positive probability
- No event has costs $\leq -1$—otherwise use of that signal would never be rationalizable

**Structural stability**

- No two events have identical probability
- No two signals have identical costs
- All signals have costs strictly $> -1$
example 1: more signals than events

- \((n, m) = (2, 3)\)
- \(\vec{e} = \langle .6, .4 \rangle\)
- \(\vec{c} = \langle 0, -.1, -.4 \rangle\)
- one possible Nash equilibrium:

\[
P^x = \begin{pmatrix} .3 & .3 & 0 \\ .3 & 0 & .1 \end{pmatrix} \quad Q^x = \begin{pmatrix} .9 & .1 \\ .9 & -.1 \\ -.9 & .1 \end{pmatrix}
\]
example 2: more events than signals

- \((n,m) = (3, 2)\)
- \(\vec{e} = \langle .5, .3, .2 \rangle\)
- \(\vec{c} = \langle 0, -.1 \rangle\)
- Nash equilibrium:

\[
P^x = \begin{pmatrix} .5 & 0 \\ .1 & .2 \\ 0 & .2 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 & 0 \\ -.1 & 0 & .8 \end{pmatrix}
\]
Example 3: a strict Nash equilibrium

- strict equilibria:
  - \( n = m \)
  - bijection between events and signals
  - ESSs are exactly the strict NE

- \( \vec{e} = \langle .75, .25 \rangle \)

- \( \vec{c} = \langle 0, -.1 \rangle \)

\[
\begin{align*}
P^{x_1} & = \begin{pmatrix} .75 & 0 \\ 0 & .25 \end{pmatrix} & Q^{x_1} & = \begin{pmatrix} 1 & 0 \\ -1 & .9 \end{pmatrix} \\
P^{x_2} & = \begin{pmatrix} 0 & .75 \\ .25 & 0 \end{pmatrix} & Q^{x_2} & = \begin{pmatrix} 0 & 1 \\ .9 & -1 \end{pmatrix}
\end{align*}
\]
Neutral stability

Definition (Neutral stability)

The (possibly mixed) strategy profile $x^*$ is *neutral stability* iff

1. $\forall y : u(x^*, x^*) \geq u(y, x^*)$, and
2. $\forall y : \text{if } u(y, x^*) = u(x^*, x^*), \text{ then } u(x^*, y) \geq u(y, y)$. 
Examples

example 4: a neutrally stable state for the previous game

\[ P^x = \begin{pmatrix} 0.75 & 0 \\ 0.25 & 0 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 \\ \alpha - 1 & 0.9 - \alpha \end{pmatrix} \]

for \( \alpha \in (0.9, 1] \).
example 5: an unstable equilibrium

\[ P^x = \begin{pmatrix} .75 & 0 \\ .25 & 0 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 \\ .8 & 0 \end{pmatrix} \]
Evolutionary stability

Observation

If $n = m$, $x$ is an ESS if and only if $S^x$ is a permutation matrix and $R^x$ its transpose.

Theorem

$x$ is an ESS if and only if

1. $m \leq n$,
2. the first column of $P^x$ has $n - m + 1$ positive entries,
3. each other column of $P^x$ has exactly one positive entry, and
4. $q^x_{ji} = 1 + c_j$ iff $i = \min\{i': p^x_{i'j} > 0\}$, otherwise $q^x_{ji} = c_j$. 

Evolutionary stability

an ESS with $m < n$

$$P^x = \begin{pmatrix} .5 & 0 \\ .3 & 0 \\ 0 & .2 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & .9 \end{pmatrix}$$
Evolutionary stability

Evolutionarily stable sets

- proposed in Thomas (1985)
- generalization of ESSet
- set of Nash equilibria that is, as a whole, protected against invasions by mutants

Definition

A set $A$ of symmetric Nash equilibria is an evolutionarily stable set (ESSet) if, for all $x^* \in A$, $u(x^*, x) > u(x, x)$ whenever $u(x, x^*) = u(x^*, x^*)$ and $x \not\in A$. 
Evolutionary stability

a non-singleton ESSet

\[ \left\{ x : \begin{pmatrix} .8 & 0 & 0 \\ 0 & .2 & 0 \end{pmatrix}, \quad Q^x = \begin{pmatrix} 1 & 0 \\ -1 & .9 \\ \alpha - .2 & .8 - \alpha \end{pmatrix} \quad \& \quad \alpha \in [0, 1] \right\} \]
Theorem

A set of strategies $A$ is an ESSet iff for each $x \in A$, $x$ is an ESS or

1. $m > n$,
2. the restriction of $P^x$ to the first $n$ columns and the restriction of $Q^x$ to the first $n$ rows form an ESS, and
3. for each $y$ such that $P^x = P^y$, and $Q^x$ and $Q^y$ agree on the first $n$ rows: $y \in A.$
Neutral stability

**Theorem**

$x$ is a NSS if and only if it is a Nash equilibrium and $Q^x$ does not contain multiple column maxima.

**Observation**

If $m, n \geq 2$, there is always at least one NSS that is not element of an ESSet.
Some facts

- In symmetrized asymmetric games:
  - The ESSs are exactly the asymptotically stable rest points under the replicator dynamics,
  - The ESSets are exactly the asymptotically stable sets of rest points under the replicator dynamics (Cressman, 2003)

- In doubly symmetric games,
  - The neutrally stable states are exactly the Lyapunov stable rest points (Thomas, 1985; Bomze and Weibull, 1995; Bomze, 2002)
Lemma

Let $x^*$ be a NSS that is not an ESS. There is some $\epsilon > 0$ such that for each Nash equilibrium $y$ with $\|x - y\| < \epsilon$,

1. $y$ is itself neutrally stable, and
2. for each $\alpha \in [0, 1]$, $\alpha x^* + (1 - \alpha)y$ is neutrally stable.
Dynamic stability

Theorem

Each NSS $x$ has some non-null environment $A$ such that each interior point in $A$ converges to some neutrally stable equilibrium $y$ under the replicator dynamics that belongs to the same continuum of NSSs as $x$. 
Dynamic stability

**sketch of proof**

(proof inspired by Pawlowitsch, 2006)

- suppose \( x \) is an NSS
- then \( x \) is Lyapunov stable
- for each environment \( U \) of \( x \), every interior point in \( U \) converges to some Nash equilibrium (Hofbauer and Sigmund, 1998; Akin and Hofbauer, 1982)
- hence almost every point in some environment \( A \) of \( x \) converges to some NSS that belongs to the same continuum of NSSs as \( x \)
Corollary

The set of Nash equilibria that do not belong to any ESS set attracts a positive measure of the state space.
Dynamic stability

Theorem

Given any strategy profile $x_1$, there is a finite sequence of profiles $(x_i)_{i \leq n}$ for some $n \in \mathbb{N}$ such that

1. there is an ESSet $E$ such that $x_n \in E$, and
2. $u(x_{i+1}, x_i) \geq u(x_i, x_i) \quad \forall i < n$. 

Conclusion

in a nutshell

- evolutionary stability: 1-1 map between min($m, n$)-many events and signals
- if $n > m$, excess events are expressed by cheapest signal
- neutral stability: some signals may remain unused, even if they would be useful
- natural selection alone does not suffice to guarantee convergence to evolutionary stability (= local maximum of average utility)
- combination of natural selection and drift does guarantee convergence to some ESSet


