2

Modal Logic

2.1 Why modality is important to logic and to semantics

Why are we so interested in the semantics of modal words? An easy answer for a linguist to give might be that we’re interested in the semantics of all kinds of words, and we might as well attend to the modal ones now rather than later. From this perspective, we could have just as well begun with words expressing family relations or describing protein sources. Surely there are perspectives on language according to which these portions of the vocabulary have as much of a claim on our attention as the modal expressions, but such perspectives do not motivate the vast majority of contemporary work in semantics. Instead, semanticists tend to think that modality is really very important. Why?

I like cute animals: pandas, pigs, praying mantises. But by studying the meanings of the words panda, pig, and praying mantis, it’s unlikely I’ll achieve even a glimmer of the pleasure I feel in learning about them, or better yet watching them. I’d better leave my interest in animals as a hobby separate from my linguistics.

I like to know things. So when I learn something new, I like to figure out what further consequences I can deduce, so that I can learn even more true things. And moreover, when you try to convince me of something, I want to evaluate the reasons you offer me for believing it. Phrases like the following are likely to come up:

(3) So this implies that dogs are mammals.

(4) No, that is impossible, and here’s why . . .

My knowledge of English tells me that if I am convinced that $X$, and I am convinced that $X$ implies $Y$, I can know more of the truth by believing $Y$ too. However, sadly, I don’t always know when $X$ implies $Y$, as opposed to when it just seems like $X$ implies $Y$. The number of mistakes I’ve made in life tells me that my judgment of such things is
flawed. It would be a good idea to study the concept of implication, and likewise for the concepts expressed by *must, impossible* and similar words.

I also want to do the right thing. Lots of people tell me what is the right thing to do in various situations, and they say things like:

5. Since you are hungry, what you ought to do is eat some lunch.

6. The teachings of the one we revere tell us that you are permitted to study linguistics.

7. Don’t complain!

Even with lots of good advice about what is the right thing to do, I sometimes get confused. Perhaps I must do X and I ought to do Y, but I don’t have the time to do both! It would be a good idea to better understand how to figure out the right thing to do in the midst of complex moral principles.

**Logic** is the study of systems of reasoning. **Modal logic** is the area of logic which specifically focuses on reasoning involving the concepts of necessity and possibility. As I use the term, modal logic includes the logics of all modal concepts, and so includes deontic logic (the logic of obligation and permission) and epistemic logic (the logic of knowledge), among others. One can also use “modal logic” in a narrow way which leaves out deontic modality, epistemic modality, and many other linguistically interesting types; on the narrow conception, modal logic covers the concepts of necessary and possible truth (**alethic modality**).¹ Through the development of modal logic, scholars have been able to better understand these philosophically important concepts which are both difficult to employ correctly and difficult to analyze clearly and objectively.

We’ve seen why logicians have been interested enough in modality to develop a logic of it, and in a moment we’ll see what modal logic has to contribute to the linguistic study of modality. But note that modal logic is not by any means the same thing as the linguistic analysis of modality. Modal logic is concerned with better understanding the concepts of implication, necessity, obligation, and the like, especially as they occur

¹ It can be hard to determine precisely which uses of modals are supposed to count as alethic, but if you put in the word *true*, as in *It is necessarily true that . . .*, the alethic reading tends to stand out. The motivation for the narrow conception may be an assumption that alethic modality is the most basic type of modality, in terms of which the other varieties may be defined, rather than just one type among many.
in patterns of reasoning. It’s not about the meanings of the natural language expressions like must, possible, and ought. In fact, in doing logic we often try to forget about the words we normally use to express these concepts, since doing so allows us to better focus on the system of reasoning itself. In a practice akin to using $F$ in physics to stand for a quantity related to—but ultimately different from—our everyday concept of force, in modal logic we use symbols as well. Commonly we write $\Box$ (or sometimes $L$) when exploring reasoning with must, ought, and necessary (among others), and we use $\Diamond$ (or $M$) when exploring reasoning with might, may, and possible (among others). Once begun, the study of $\Box$ and $\Diamond$ takes on a life of its own, and logicians get to know a great deal about what kinds $\Box$’s and $\Diamond$’s there can be, without any pretense that this investigation has anything to do with the semantics of human language. Like logic in general, modal logic has a side which is rather similar to abstract mathematics.

Though modal logic is not the same thing as the linguistics of modal expressions, the deeper understanding which modal logic has given us of the concepts of implication, necessity, obligation, and the like, shows us why modal expressions demand the linguists’ attention: These concepts both have practical importance and are very different from non-modal concepts. (The remainder of this chapter will be devoted showing why this is so.) Therefore, a semantic theory which does not attend to modality will be radically simpler than one which does, and so will provide a much less accurate overall picture of the nature of meaning in human language. In this way, modal vocabulary is very different from the cute animal vocabulary; if we leave out the cute animals, we’ll miss some interesting details, but the parts of the theory which explain how we talk about cute people and ugly animals will presumably be pretty similar to what we’d need to take care of the cute animals as well. This is why modality is one of the most important topics in semantics, and cute animal vocabulary is not.

2.2 Some basic ideas from modal logic

2.2.1 Frames and models

Here is a certain kind of ant which identifies its sex by smell. Every ant gives off either a female smell or a male smell at all times. Suppose that a certain ant wants to know if it is a female or a male. All it can do is smell. But since it works in the nest and is surrounded by many other
ants at all times, it can’t just smell itself; the smells of all the nearby ants are mixed together. However, the ant does know that its own scent is among all of the ones it smells mixed together. Therefore it knows (8):²

(8) If every ant which I smell is female, then I am female.

Not only does our ant know (8); every ant is in a position to know it, provided that he or she realizes that its own scent is among the scents that it can smell.

This ant also wants to know if it has a white spot on its back, but since it cannot see its own back, a statement similar to (8), namely (9), is not something it can be certain of.

(9) If every ant which I see has a white spot on its back, then I have a white spot on my back.

There’s a crucial difference between what the ant can figure out based on its sense of smell and what it can figure out based on its sense of sight. Because it can smell itself, it knows (8), but because it cannot see its own back, it cannot know (9). We say that, for ants, smell is reflexive: one can always smell oneself. But for ants, seeing-one’s-back is not reflexive.

Examples (8)–(9) illustrate a case where an ant can know more by its sense of smell than its sense of sight, but this does not mean that smell is always better. Since ants have compound eyes, they can see in all directions at once. This means that if one ant sees another, the second one sees the first as well. We say that ant vision is symmetrical: if ant 1 sees ant 2, then ant 2 sees ant 1. Because of this, our ant knows (10). (Let’s assume that the spots are so big that, if you can see a particular ant at all, you can see whether it has a spot on its back.)

(10) If I have a white spot on my back, then every ant which I see sees an ant with a white spot on its back.

The symmetry of ant vision is what lets our ant be certain of (10). As a result, if the ant asks around and finds that not every ant it sees itself sees an ant with a white spot on its back, it can conclude that it does not have a white spot on its back.

² This way of introducing the basic ideas of modal semantics was inspired by Hughes and Cresswell’s (1996) modal games. The semantics for modal logic on which the games are based is derived from the work of many scholars, including Kanger (1957), Bayert (1958), Kripke (1959), Kripke (1963), Montague (1960), and Hintikka (1961), and is especially connected with Kripke’s name.
Let us review what we have seen in a more abstract format. If we are considering only the sense of smell, then for any ant (11) is true and (12) is false; in terms of vision, in contrast, for any ant (11) is false and (12) is true. (The variable P can stand for any property at all that an ant might have.)

(11) If every ant which I sense is P, then I am P.

(12) If I am P, then every ant which I sense senses an ant which is P.

Even more abstractly, all of this has nothing really crucial to do with ants, of course. What’s essential is only that the relation between ants based on smell is reflexive and that based on vision is symmetrical. So we can say that (11) is valid on a reflexive relation and (12) valid on a symmetrical relation.

Modal logic allows us to build a general, abstract theory of the connection between notions like reflexivity and symmetry and the validity of statements like (11) and (12). In order to see how this works, let us begin by defining a modal logic language $\mathcal{MLL}$.

(13) Definition of Modal Logic Language $\mathcal{MLL}$:

1. Atomic sentences:
   An infinite number of variables are sentences of $\mathcal{MLL}$:
   \[ p, q, r, \ldots \]

2. Negation:
   If $\alpha$ is a sentence of $\mathcal{MLL}$, then so is $\neg \alpha$.

3. Conjunction, Disjunction, and Conditionals:
   If $\alpha$ and $\beta$ are sentences of $\mathcal{MLL}$, then so are $\alpha \land \beta$, $\alpha \lor \beta$, and $\alpha \rightarrow \beta$.

4. Necessity and Possibility:
   If $\alpha$ is a sentence of $\mathcal{MLL}$, then so are $\Box \alpha$ and $\Diamond \alpha$.

Actually we don’t need quite so many symbols in the language. The ways that logicians intend to use $\mathcal{MLL}$ means that some of them can be defined in terms of the others. For example, we can take just $\neg$, $\lor$, and $\Diamond$ as basic, and define the others as follows:

1. $\alpha \land \beta = \neg (\neg \alpha \lor \neg \beta)$
2. $\alpha \rightarrow \beta = (\neg \alpha \lor \beta)$
3. $\Box \beta = \neg \Diamond \neg \beta$

The last point is the only one that has specifically to do with modality. Intuitively, it embodies the assumption that something is necessary if and only if it’s not possible that it’s false.
Next we define two concepts which are essential for providing a semantics to MLL: **frames** and **models**.

(14) A frame is a pair \((W, R)\) consisting of a set \(W\) and a relation \(R\) on \(W\).

For example:

(15) 1. \(W\) is the set of three ants \(\{\text{ant } 1, \text{ant } 2, \text{ant } 3\}\).

2. \(R\) is the “smells” relation such that:
   - (a) Ant 1 smells itself.
   - (b) Ant 1 smells ant 2.
   - (c) Ant 1 smells ant 3.
   - (d) Ant 2 smells itself.
   - (e) Ant 2 smells ant 1.
   - (f) Ant 3 smells itself.
   - (g) There are no other cases of an ant smelling another.

Above we learned that ant-smelling is reflexive, and this is formalized in terms of the notion of frame. This is an example of a reflexive frame, since for each member of \(W\), the relation \(R\) holds between that thing and itself. Given what we said about ant-smell, every frame which aims to represent what ants can learn by smelling should be reflexive.

The members of \(W\) can be anything at all, but typically they are called “possible worlds,” and so a more typical description of a frame (one which is identical to (15) as far as logical properties go) might be as follows:

(16) \(F = (W, R)\) as follows:

1. \(W = \{w_1, w_2, w_3\}\)

2. \(R\) is the smallest\(^3\) relation on \(W\) such that:
   - (a) \(R(w_1, w_1)\)
   - (b) \(R(w_1, w_2)\)
   - (c) \(R(w_1, w_3)\)
   - (d) \(R(w_2, w_2)\)
   - (e) \(R(w_2, w_1)\)
   - (f) \(R(w_3, w_3)\)

\(^3\) This means that for no other \(x, y \in W\) is it the case that \(R(x, y)\).

In what follows, I’ll write \(R(x, y)\) to say that \(x\) stands in the \(R\) relation to \(y\). In logic it is customary to identify a relation with its extension, so that \(R(x, y)\) could also be written as \((x, y) \in R\). I find the \(R(x, y)\) notation easier to use, especially in running text, and don’t mean to imply that \(R(x, y)\) is an expression in some special logical language.
We’ll come back to this concept of possible world shortly; for the time being we’ll stick with ants, so that we can focus on the nature of the logic without getting distracted by metaphysics. Once we understand the logic well, we will be able to see why it makes sense to construct the frames out of possible worlds.

Since \( R(\text{ant } 1, \text{ant } 1) \), \( R(\text{ant } 2, \text{ant } 2) \), and \( R(\text{ant } 3, \text{ant } 3) \), the frame above is reflexive. A symmetrical relation (like ant vision) can give rise to a symmetrical frame, and other types of relations, based on other types of frames, will be important in what follows. Here are definitions of some types of frames:

1. **Reflexive frame**: \( \langle W, R \rangle \) is a reflexive frame if (and only if) \( R(w, w) \) for every \( w \in W \).
2. **Symmetrical frame**: \( \langle W, R \rangle \) is a symmetrical frame if for every \( w \) and \( w' \in W \), if \( R(w, w') \), then \( R(w', w) \).
3. **Serial frame**: \( \langle W, R \rangle \) is a serial frame if for every \( w \in W \), there is a \( w' \in W \) such that \( R(w, w') \).
4. **Transitive frame**: \( \langle W, R \rangle \) is a transitive frame if for every \( w, w' \) and \( w'' \in W \), if \( R(w, w') \) and \( R(w', w'') \), then \( R(w, w'') \).
5. **Equivalence frame**: \( \langle W, R \rangle \) is an equivalence frame if it is a reflexive, symmetrical, and transitive frame.

Now we define a model.

(17) A model is a pair \( \langle F, V \rangle \) such that \( F \) is a frame and \( V \) is a function which associates each pair of a member of \( W \) and an atomic sentence of MLL with the value 1 or the value 0.

The function \( V \) is called a **valuation function**. Thus we can say that a model is a pair of a frame and a valuation function. The value 1 conventionally stands for “true”, so that if \( V(x, p) = 1 \), this is to say that \( p \) is being treated as true for \( x \) in the model. Likewise, the value 0 conventionally stands for “false”. Here’s an example of a model:

(18) 1. \( M = \langle F, V \rangle \) such that:
   (a) \( F \) is as in (15)
   (b) \( V \) is as follows:
      i. \( V(\text{ant } 1, p) = 1 \)
      ii. \( V(\text{ant } 1, q) = 1 \)
      iii. \( V(\text{ant } 2, p) = 1 \)
      iv. for all other \( x \in W \) and atomic sentences \( \phi \),
         \( V(x, \phi) = 0 \).

\(^4\) From now on, I’ll use the common abbreviation “iff” for “if and only if.”
Figure 2.1. The model (18)

Figure 2.1 illustrates this model graphically. Ants are numbered as 1, 2, and 3 clockwise from the upper left. An arrow from one ant to another means that the first can smell the second. For example, ant 1 can smell itself and the other two ants. This has a reflexive frame (based on the reflexive relation “smells”) since each ant has an arrow turning back to itself. The list of letters next to an ant represents the atomic sentences which are true for that ant. Perhaps $p$ is “this ant is female” and $q$ is “this ant has a white spot on its back.” In that case, ant 1 is female and has a white spot on its back. Notice that, in this model, each ant can truthfully say (8), “if every ant which I can smell is female, then I am female.” (Ant 1 smells two female ants, itself and ant 2, and one male ant, ant 3; because not every ant it smells is female, it can’t decide whether it’s female or not, and so on the basis of the definition of $\rightarrow$, (8) comes out as true for ant 1. Ants 2 only smells female ants, and is itself female, so (8) is true for ant 2. Ant 3 smells one male ant, itself, and again (8) is true for this ant.) Of course this is as expected, because we saw earlier that this sentence should be true whenever the frame is reflexive.

Our next job is to give a formal characterization of what it is for an ant to be able to say truthfully some sentence $\alpha$. We begin by starting with the simplest sentences, the atomic sentences, and then determining whether more complex sentences are true for each ant based on simpler ones. If $\alpha$ is an atomic sentence, the ant can say $\alpha$ truthfully iff $V$ assigns 1 to the pair of that ant and $\alpha$. For example, $V(\text{ant 1, } p) = 1$, so ant 1 can truthfully say “$p$”. Next we want to allow for more complex sentences; for example, ant 1 can truthfully say “$p$ and $q$”, or as we write it with logical notation ($p \land q$), since she can truthfully say both “$p$” and “$q$”. What we want to do is take $V$ and extend it from just the atomic sentences to all of the sentences of $\text{MLL}$. We will write $\llbracket \alpha \rrbracket_w^V$ to indicate the truth value (1 or 0) of $\alpha$ in the world $w$ on the
model $M$.\(^5\) In the ant model, $[[ p ]]^{\text{ant}}_{1,M} = 1$ means that ant 1 in modal $M$ can truthfully say “$p$”.

(19) For any model $M = \langle (W, R), V \rangle$ and any $w \in W$, $[[ a ]]^w,M = 1$ iff one of the following conditions hold (and otherwise $[[ a ]]^w,M = 0$):

1. $a$ is an atomic sentence and $V(w, a) = 1$
   (For atomic sentences, $[[ a ]]^w,M = V(w, a)$.)
2. $a$ is of the form $\neg \beta$ and $[[ \beta ]]^w,M = 0$
   ($\neg \beta$ is true iff $\beta$ is false.)
3. $a$ is of the form $(\beta \land \gamma)$ and $[[ \beta ]]^w,M = 1$ and $[[ \gamma ]]^w,M = 1$
   ($(\beta \land \gamma)$ is true iff both $\beta$ and $\gamma$ are true.)
4. $a$ is of the form $(\beta \lor \gamma)$ and $[[ \beta ]]^w,M = 1$ or $[[ \gamma ]]^w,M = 1$
   ($(\beta \lor \gamma)$ is true iff $\beta$ or $\gamma$ or both are true.)
5. $a$ is of the form $(\beta \rightarrow \gamma)$ and $[[ \beta ]]^w,M = 0$ or $[[ \gamma ]]^w,M = 1$
   ($(\beta \rightarrow \gamma)$ is true iff either $\beta$ is false or $\gamma$ is true.)
6. $a$ is of the form $\Box \beta$ and for all $v$ such that $R(w, v)$, $[[ \beta ]]^v,M = 1$
   ($\Box \beta$ is true iff $\beta$ is true in all members of $W$ accessible from $w$.)
7. $a$ is of the form $\Diamond \beta$ and for some $v$ such that $R(w, v)$, $[[ \beta ]]^v,M = 1$
   ($\Diamond \beta$ is true iff $\beta$ is true in some member of $W$ accessible from $w$.)

Let me make some comments about the definitions of $\rightarrow$, $\Box$, and $\Diamond$.

We say that $(\beta \rightarrow \gamma)$ is true iff either $\beta$ is false or $\gamma$ is true; this is something close to the meaning of if–then, but it’s clearly not the same as if–then. Don’t worry about this problem; it’s the best we can do within this simple logical system, and is the standard definition of $\rightarrow$. It is not how linguists typically analyze the meaning of if–then within their richer semantic theories. An elementary logic textbook will provide a justification for defining the meaning of $\rightarrow$ this way, but as we are only mining modal logic for the purposes of learning some things to apply to natural language semantics, it’s not worthwhile going into it here. We’ll discuss conditionals from different perspectives in Sections 3.1 and 5.2.

The descriptions of the definitions of $\Box$ and $\Diamond$ make use of the term “accessible from.” Accessibility holds between members of $W$ based on

\(^5\) A common notation for logicians would be $w \models_{M,a}$ or something similar. The $[[ ]]^w,M$ used here is the kind of notation used by linguists in developing theories of the semantics of natural language, and I use it here for the sake of consistency with later sections.
the relation $R$. That is, $v$ is said to be accessible from $w$ iff $R(w, v)$. In fact, $R$ is often called an accessibility relation. In the case of our ant model, for example, ant 2 is accessible from ant 1 because ant 1 smells ant 2, that is $R(\text{ant 1}, \text{ant 2})$.

Under these definitions, $[[\Box p]]^w_M$ means that $p$ is true in all members of $W$ accessible from $w$. In the case of our ant model, $[[\Box p]]^\text{ant 1}_M$ means “$p$ is true for every ant that ant 1 can smell.” Let’s adopt the convention that the pronouns I and me refer to the ant who is saying the sentence. For example, when we are computing $[[\Box \phi]]^\text{ant 1}_M$, these pronouns refer to ant 1. Then we can say that $\Box p$ means “$p$ is true for every ant I can smell.” Likewise $\Diamond p$ means that “$p$ is true for some ant I can smell.” The following formula expresses the meaning “if $p$ is true for every ant I can smell, it’s true for me”:

$$\Box p \rightarrow p$$

(20) is the MLL counterpart of (11), and it is commonly called T. (The reason why it is an important enough formula to warrant its own one-letter name will be explained in Section 2.2.4.) It is true for all three ants in our model, as is expected because our model has a reflexive frame.

The MLL counterpart of (12) is (21), called B:

$$p \rightarrow \Box \Diamond p$$

(21) This formula expresses the meaning “if $p$ is true for me, then every ant I can smell can smell an ant for which $p$ is true.” B is not true for ant 1 in our “smelly” model, since $p$ is true for ant 1, but ant 1 can smell ant 3, and ant 3 cannot smell any ant for which $p$ is true. In other words, ant 1 is female, but not every ant that ant 1 smells can smell a female ant. It is not surprising that (21) isn’t true on the “smelly” frame, since the reflexivity of smell did not guarantee the truth of (12). However, we would expect it to be true on a symmetrical “seeing” frame.

### 2.2.2 Validity

We can now go on to define different version of the crucial logical property of validity. The central idea of validity is that a sentence is valid iff it must be true on the basis of the language’s syntax and semantics. Because MLL is built entirely from vocabulary which is either logical in nature ($\land$, $\Box$, etc.) or is devoid of specific content ($p$’s

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6 If we think about a natural language, where individual words have interesting meanings, we might distinguish those sentences which must be true on the basis of syntax and compositional semantics from the broader class of those which are true on the basis of
meaning is completely arbitrary), the valid sentences will all be valid due to logical properties. For example, \( \neg(p \land \neg p) \) is valid, because \( p \) cannot be both true and not true, no matter what \( p \) itself means (and this intuitive fact is guaranteed by our definitions). What we are primarily interested in here are those sentences which are valid because of the meanings we assign to \( \square \) and \( \Diamond \).

First we will define validity with respect to a particular model, and then we will define it with respect to all models which share the same frame \( F \).

(22) A sentence \( \alpha \) is \textit{valid in a model} \( M = \langle \langle W, R \rangle, V \rangle \) iff \( \llbracket \alpha \rrbracket^w_M = 1 \) for every \( w \in W \).

(23) A sentence \( \alpha \) is \textit{valid on a frame} \( F \) iff, for every valuation function \( V \), \( \alpha \) is valid in the model \( M = \langle F, V \rangle \).

(22) says that \( \alpha \) is true for every member of \( W \), given a particular model. That is, every ant agrees that \( \alpha \) is true. (23) is more general; it says that \( \alpha \) is true no matter which atomic sentences are true for each ant, just so long as we keep the ants and the relations among them the same. That is, you can go into Figure 2.1 and changes the \( p \)'s and \( q \)'s however you want (but don’t introduce or remove ants, or change the arrows), and \( \alpha \) will still be true for every ant. It should be clear that more sentences can be valid in a particular model than are valid on the frame of that model: It might just happen that in the model, \( p \) is true for every ant, and so \( p \) is valid on that model. But once we change the valuation function (keeping the frame the same), \( p \) is false for one or more of the ants. For this reason, \( p \) and other atomic sentences are never valid on a frame. But some complex formulas are going to be valid on a frame. For example, not only is \( T = (\neg 20) : \square p \rightarrow p \) valid in the model \( M \) described by Figure 2.1, it’s valid in the frame \( F \) which we used to define \( M \).

Finally, we can consider what happens if we allow the frames, i.e. the ants and the relations among them, to vary. The most general concept is \( K \)-validity:

(24) A sentence \( \alpha \) is \textit{K–valid} iff it is valid on every frame.

syntax and semantics (both lexical and compositional). The first group we might call valid and the latter, \textit{tautologies}. However, in the logic we’re working with here, this distinction is not important.
We can also get various more refined notions by considering some subset of all the possible frames. Here are a number of the important versions of validity:

(25) (a) A sentence $a$ is T-valid iff it is valid on every reflexive frame.
    (b) A sentence $a$ is B-valid iff it is valid on every symmetrical frame.
    (c) A sentence $a$ is D-valid iff it is valid on every serial frame.
    (d) A sentence $a$ is S4-valid iff it is valid on every transitive frame.
    (e) A sentence $a$ is S5-valid iff it is valid on every equivalence frame.

We already know that T, ($\Box p \rightarrow p$) is T-valid and B ($p \rightarrow \Box \Diamond p$) is B-valid. We’ll learn more about why the various kinds of validity are important in Section 2.2.4.

2.2.3 Possible worlds

As far as logic is concerned, it does not matter what is in the set $W$. It could be ants, it could be numbers, it could be sentences, it could be a mix of different things. All that matters is how many things we have in $W$ and what relations among them are established by $R$, and for this reason modal logic can be useful in analyzing any phenomenon which can be modeled in terms of a set of objects and relations among them. Nevertheless, modal logic was originally invented in order to develop a theory of reasoning using modal concepts, and in fact the logic we’ve developed does a decent job of this.

Consider the concept of T-validity. We have seen that T is T-valid.

$$T: \Box p \rightarrow p$$

Now consider the following sentences with must which have the form of T:

(26) (a) If it must be raining, then it is raining.
    (b) If Mary must tell the truth, then she will tell the truth.

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7 The names for the various kinds of validity come from the links with axiomatic systems of the same names discussed in Section 2.2.4. However, the names of the axiomatic systems themselves are not based on anything easy to remember; they just come from various important works in the history of modal logic. See a good modal logic textbook like Hughes and Cresswell (1996) or Blackburn et al. (2001) for more of the historical and bibliographical details.
In (26a), *must* has an epistemic meaning, and intuitively the sentence has to be true. In contrast, in (26b), *must* has a deontic meaning, and the sentence is not necessarily true. This suggests that concept of T-validity would not be appropriate for giving a logic of modals in their deontic meanings, but it might be useful for giving a logic of epistemic modals.

Now consider the following sentence, commonly called D, which is both T-valid and D-valid:

(27) \[D: \Box p \rightarrow \Diamond p\]

Here are some English sentences with the form of D, the first containing epistemic modals and the second deontic modals:

(28) (a) If it must be raining, then it might be raining.
     (b) If Mary must tell the truth, then she may tell the truth.

Both of these are intuitively true. (In (28b), we are concerned with the reading where both *must* and *may* are deontic; in particular, *may* means “is permitted to.” It also has a reading where *may* means “is possible that.”) The fact that (28a) has the form of a T–valid sentence further supports the use of the concept of T–validity to create a logic for epistemic modals; the fact that (28b) is D–valid suggests that D–validity might be useful in giving constructing a logic of deontic modality.

Modal logic captures many of the logical properties of modal expressions. Of course in a way this is not surprising, since this is what it was designed to do, but why are these abstract concepts defined in terms of frames and models part of the picture? For example, why does the notion of T–validity seem to capture some important aspects of the meaning epistemic modals? The reason is that we can understand modal expressions to invoke particular frames which have the general properties of reflexivity, seriality, etc. These frames will have a set \(W\) and an accessibility relation \(R\) just like any frames, but the nature of \(W\) and \(R\) will be specified in greater detail. In particular \(W\) will be a set of possible worlds and \(R\) will be defined in terms of linguistically relevant concepts like knowledge or rules.

The notion of possible world goes back to the work of Leibniz and plays an important role in modern logic and semantics. A possible world is a complete way that the universe could be throughout its history. For example, our universe (the “actual world”) is a possible world. There are other possible worlds which are like our world except that some minor detail is changed; perhaps it’s .0001 degree colder in
London today. Other worlds involve major changes. Perhaps dinosaurs never went extinct; there are lots of different possible worlds realizing this general scenario. Still others never had much to do with our world at all. For example, there are possible worlds in which only two elementary particles exist, and time is circular, with those two particles orbiting each other once for each cycle of time. Perhaps some possible worlds are not even conceivable by humans, but happily if there are such worlds, they can safely be ignored by linguists.

Given the notion of possible worlds, consider the following type of frame. In this frame $R$ is defined in terms of the someone's knowledge. We have $R(w, w')$ if everything that some individual $i$ knows in $w$ is true in $w'$. Let’s call $R$ an epistemic accessibility relation and $F$ an epistemic frame:

\[(29)\quad \text{Epistemic frame}\]

\[F = \langle W, R \rangle \text{ is an epistemic frame iff for some individual } i:\]

- $W =$ the set of possible worlds conceivable by humans.
- $R =$ the relation which holds between two worlds $w$ and $w'$ iff everything which $i$ knows in $w$ is also true in $w'$.

An epistemic frame is reflexive, since it’s a property of knowledge that if $i$ knows $p$ in $w$, then $p$ is true in $w$. For example, if I know that it’s raining right now, then it is indeed raining right now. It is often said that knowledge is properly justified true belief (but see Gettier 1963), and the fact that what’s known must be true implies reflexivity. We can say that a sentence is “epistemically valid” iff it is true on all epistemic frames. Since all epistemic frames are reflexive, anything which is T–valid is also epistemically valid. This justifies using modal logic to try to better understand epistemic modality. We can also ask the converse question: are all epistemically valid sentences also T–valid, i.e. are the epistemically valid sentences and the T–valid sentences identical? My personal answer is: I’m not sure.

We can define another important class of accessibility relations, the deontic accessibility relations. A deontic accessibility relation is one defined in terms of a particular system of rules. We have $R(w, w')$ iff all of the rules of $w$ are followed in $w'$. For example, in our world the rule “No murder!” holds, and so we have $R(\text{our world}, w')$ only if there is no murder in $w'$. More generally, $R(w, w')$ means that $w'$ is a perfect world from the perspective of the rules of $w$.

Based on the concept of a deontic accessibility relation, we can define a deontic frame:
(30) **Deontic frame**

\[ F = (W, R) \] is a deontic frame if for some system of rules \( r \):

- \( W \) is the set of possible worlds conceivable by humans.
- \( R \) is the relation which holds between two worlds \( w \) and \( w' \) if all of the rules which \( r \) establishes in \( w \) are followed in \( w' \).

Note that in this definition, \( r \) is not just a set of rules. It is called a "system of rules," and it associates each world in \( W \) with a set of rules. Different worlds may be associated with different rules. For example, let \( r \) represent the moral precepts which humans should live by. Suppose we believe that a benevolent God is responsible for the nature of morality. God created all of the possible worlds, and associated a moral order with each one. The moral orders may differ somewhat from world to world, since the creatures living in each world may have different natures and so need different rules.

A sentence is deontically valid if it is true on every deontic frame. A deontic frame is not necessarily reflexive, as we can see from the fact that, though "No murder!" is a moral precept of our world, there is nevertheless murder. However, we might require that every deontic frame is serial. By invoking seriality, we would be saying that it’s no good to have an inconsistent set of rules, a set all of which cannot be satisfied together. Of course in reality, we may find ourselves in a situation with contradictory requirements (a “Catch-22”). According to the idea that deontic frames are always serial, these requirements do not constitute a proper system of rules.

Let us assume that a deontic frame is defined to be serial. Given seriality, we know that any sentence which is D–valid is also deontically valid; this justifies using modal logic to better understand deontic modality. Again, it’s not clear whether every deontically valid sentence is D–valid, that is whether D–validity is precisely the same concept as deontic validity.

If we interpret \( \square p \) on an epistemic frame, it will give a reasonably good analysis of the sentence “In light of \( i \)'s knowledge, it must be that \( p \)” or “\( i \) knows that \( p \).” Likewise, if we interpret \( \square p \) on a deontic frame, it goes a fair way towards providing an analysis of “In light of \( r \), it must be that \( p \)” or “In light of \( r \), \( p \) is obligatory.” This success suggests that our understanding of modal sentences actually involves an understanding of alternative possible worlds. When you come to think of it, it doesn’t seem so surprising that we would have some way of talking about alternative possible worlds, given the obvious fact about
humans that we can imagine the world being different from how it is. In particular, the kinds of things we talk about with modal sentences seem intuitively to involve the alternative ways the world could be. For example, the whole point of specifying a law is that sometimes people don’t behave in accordance with it. We know how we want things to be, even though we’re quite aware that things are not that way. So the law specifies a set of possible worlds.

Let us look at these points with an example. The following law describes those worlds in which all scooter drivers wear helmets from the time the law comes into force:

(31) Henceforth, all scooter drivers will wear helmets.

Based on this law, we now want to describe our world as one in which this law holds. Clearly this doesn’t amount to our world being one of those described by (31); that is, the mere fact that we have this law doesn’t ensure that all scooter drivers will in fact wear helmets. Rather, once we’ve introduced this law, our world in one in which an alternative world can only be “how things legally should be” if it has no helmetless scooter drivers. In other words, our world is one in which the following is true:

(32) □(all scooter drivers wear helmets.)
    ≈ All scooter drivers must wear helmets.

Thus, possible worlds provide an intuitively appealing tool for bringing out the details of what (32) means, and moreover the appeal to possible worlds lets us understand the success of the □ in modeling the meaning of deontic must. For these reasons, possible worlds and modal logic have been seen as an extremely appealing basis on which to develop semantic theories of modal expressions in natural language.

2.2.4 Axiomatic systems

In Section 2.2, I presented some of the ideas from modal logic which are especially important to linguistic semantics. However, there's much more to modal logic than this, and in what follows I briefly discuss one other important aspect of modal logic. This section is designed to give readers a flavor for how the ideas from modal logic on which linguists draw fit into the broader field.

The original conception of modal logic was syntactic (e.g., Lewis 1918; Gödel 1933; see Blackburn et al. 2001 for a brief historical overview). Given a modal language like MLL, a system of axioms and
system for proving theorems from these axioms can be added, with the logic being, in effect, the set of theorems that can then be proven. For example, let’s consider the following logic K built on MLL:

(33) Axioms of K
    (a) If $\phi$ is a valid sentence of non-modal propositional logic, then $\phi$ is an axiom of K.
    (b) K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is an axiom of K.

(34) Rules of Proof
    (a) Uniform Substitution
        If $\phi$ is a theorem of K, then so is the result of replacing any of the variables, in a uniform way, with sentences of MLL.
    (b) Modus Ponens
        If both $\phi$ and $(\phi \rightarrow \psi)$ are theorems of K, then so is $\psi$.
    (c) Necessitation
        If $\phi$ is a theorem of K, then so is $\Box \phi$.

The first set of axioms in our modal logic, those given by (33a), is not specifically modal in nature. This just says that if something is valid in the language we had before adding modal operators, we want it to be a theorem of the modal language. This makes sense. For example, if we accept that “$p$ or not $p$” ($p \lor \neg p$) is obviously true (as most people do), we are entitled to assume it as we go about proving things in modal logic.

Axiom (33b) is a specifically modal axiom. This axiom is traditionally known as K (note: boldface) because what distinguishes the logic K (no boldface) from other modal logics is that it has only this as a specifically modal axiom. In other words, since we are going to compare K to other modal logics which have modal axioms besides K, we can say that K defines K. Reading $\Box$ as necessary, this axiom says that if it’s necessary that $p$ implies $q$, then if it’s necessary that $p$, it’s also necessary that $q$. Does that sound like a principle we should accept? How about if we we replace necessary with must? K does seem pretty solid, though modal logics that do not accept it have been developed. In any case, it is worthwhile to at least develop the logic that comes about if we accept K, and see if it turns out to be useful.

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8 Hughes and Cresswell (1996) provide a straightforward introduction, on which my presentation is based.
The first two rules of proof are not specifically modal, and we won’t dwell on them here. The third says that if you can prove some sentence \( a \), you can immediately prove \( \Box a \) as well. The idea here is that proving something is one way of finding out that it’s a necessary truth. More precisely, necessitation says that if you can prove something using only the axioms and rules above, it can confidently be counted as necessary. For example, \((p \lor \neg p)\) is a theorem of K (through a trivial proof, since it is an axiom by \((33a)\)). Since we’re completely sure of \((p \lor \neg p)\), we are also completely sure that \((p \lor \neg p)\) must be true, i.e. \(\Box(p \lor \neg p)\). This principle aims to capture an important connections between provability and necessity: provability implies necessity. If you’re not sure that it’s gotten the connection completely right, that’s also fine; again, it’s at least worthwhile to accept Necessitation provisionally and see where it leads us.

One can get other modal logics by using different axioms. Some of the most well-known modal logics and axioms which define them are given in Tables 2.1 and 2.2. Some of the formulas have been introduced earlier (e.g., T and B), and in a moment we’ll see why.

This plethora of axioms and systems may be bewildering at first. What does each axiom mean? Should it be accepted in general, or for

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### Table 2.1. Some important axioms for modal logics

<table>
<thead>
<tr>
<th>Axiom name</th>
<th>Axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>(\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))</td>
</tr>
<tr>
<td>T</td>
<td>(\Box p \rightarrow p)</td>
</tr>
<tr>
<td>B</td>
<td>(p \rightarrow \Box \Box p)</td>
</tr>
<tr>
<td>D</td>
<td>(\Box p \rightarrow \Diamond p)</td>
</tr>
<tr>
<td>4</td>
<td>(\Diamond p \rightarrow \Box \Diamond p)</td>
</tr>
<tr>
<td>E</td>
<td>(\Diamond p \rightarrow \Box \Diamond p)</td>
</tr>
</tbody>
</table>

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### Table 2.2. Some systems of modal logic and their axioms

<table>
<thead>
<tr>
<th>System name</th>
<th>Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>T</td>
<td>KT</td>
</tr>
<tr>
<td>B</td>
<td>KT B</td>
</tr>
<tr>
<td>D</td>
<td>KD</td>
</tr>
<tr>
<td>S4</td>
<td>KT 4</td>
</tr>
<tr>
<td>S5</td>
<td>KT E</td>
</tr>
</tbody>
</table>
some specific purpose, or not at all? Which other formulas should also be considered as potential axioms? With careful thought and the kind of practice one gets through the study of modal logic, this sense of bewilderment can be greatly reduced. For example, \( D \) is a reasonable axiom if we are developing a deontic logic. In this context, \( D \) expresses the easily accepted idea that if something is obligatory, it’s permissible. On the other hand, in deontic logic we would not want to accept \( T \), since this would amount to saying that whatever is obligatory is true.

Despite the fact that with practice one can get a much better handle on what uses each potential axiom of modal logic has, practice alone will never give a firm answer to the question of whether a particular set of axioms defines exactly the logic we want. \( K \) and \( D \) are good axioms to choose if you want a deontic logic, but exactly how much can you prove with them? Is it just the right amount, not quite enough, or too much? So long as a given logic is analyzed purely in terms of the theorems provable within some axiomatic system, we cannot characterize the logic in any way other than in terms of those theorems. That is, we can say things like “our logic proves theorem X” or “this logic proves everything that logic proves, plus some more,” but such statements don’t provide an independent sense of what all of the theorems of a given logic have in common. A related weakness of the axiomatic approach is that it provides no obvious way to show that some formula is not a theorem of a given system. In contrast, with the frame semantics we can show that something is not valid by finding an appropriate model in which it is false (as, for example, we showed that \( T \) is not \( D \)-valid).

From a practical point of view, what we’d most like to know about a given logic is whether its theorems are all of the principles we’d want to accept for some particular type of reasoning. For example, is \( D \) the correct logic for moral reasoning? This question is impossible to answer, of course, without a clear understanding of morality itself, and this isn’t something to be given by logic or linguistics. However, there are other ways of characterizing what all of the theorems of a given logic have in common which allow a much deeper understanding of the nature of the logics in Table 2.2. The most famous of these links axiomatic systems to frame semantics.

Recall the definition of \( K \)-validity in (24). This says that a sentence is \( K \)-valid if it’s valid on any frame whatsoever. It is not hard to see that \( K \) is \( K \)-valid. \( K \) says that if every world is one in which \( (p \rightarrow q) \) is true, then if every world is one in which \( p \) is true, every world is also one in
Table 2.3. Some systems of modal logic and equivalent definitions of validity in frame semantics

<table>
<thead>
<tr>
<th>System name</th>
<th>Axioms</th>
<th>Frame type</th>
<th>Validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>K</td>
<td>all</td>
<td>K-valid</td>
</tr>
<tr>
<td>T</td>
<td>KT</td>
<td>reflexive</td>
<td>T-valid</td>
</tr>
<tr>
<td>B</td>
<td>KT B</td>
<td>symmetrical</td>
<td>B-valid</td>
</tr>
<tr>
<td>D</td>
<td>KD</td>
<td>serial</td>
<td>D-valid</td>
</tr>
<tr>
<td>S4</td>
<td>KT 4</td>
<td>transitive</td>
<td>S4-valid</td>
</tr>
<tr>
<td>S5</td>
<td>KT E</td>
<td>equivalence</td>
<td>S5-valid</td>
</tr>
</tbody>
</table>

which $q$ is true. To see that this is K-valid, consider an arbitrary world $w$ and an arbitrary frame $F$. Either $\Box(p \rightarrow q)$ is true in $w$ or it’s not. If it’s not, then $K$ is true (since the definition of $\rightarrow$ makes $(\alpha \rightarrow \beta)$ true if $\alpha$ is false). If it is, then we ask whether $(\Box p \rightarrow \Box q)$ is true in $w$. Well, either $\Box p$ is true in $w$ or it’s not. If it’s not, then $\Box p \rightarrow \Box q$ is true in $w$, so overall $K$ is true. If it is, then we ask whether $\Box q$ is true in $w$ as well. It is true if $q$ is true in every world accessible from $w$. Let $v$ be an arbitrary world accessible from $w$. We already know that $\Box p$ is true in $w$, and this implies that $p$ is true in $v$. We also know that $\Box(p \rightarrow q)$ is true in $w$, and thus $p \rightarrow q$ is true in $v$ as well. If both $p$ and $(p \rightarrow q)$ are true in $v$, $q$ is true in $v$ as well. Since $v$ was an arbitrarily chosen world accessible from $w$, we know that $q$ is true in every world accessible from $w$; therefore $\Box q$ is true in $w$. We have therefore proven that $K$ itself is true in $w$. Finally, since $w$ was an arbitrary world, we know that $K$ is true in any world, i.e. that $K$ is valid on frame $F$. Since we didn’t use any particular properties of the frame to prove $K$, we could have proven $K$ to be valid in any frame whatsoever. In other words, $K$ is K-valid.

It’s easy to show that $K$ is K-valid. What’s harder to prove—but provable, as any modal logic textbook will show you—is that $K$ is precisely the set of K-valid sentences. In other words, the set of theorems provable with $K$ equals the set of sentences valid on any frame whatsoever. Technically, we say that $K$ is sound and complete with respect to the set of all frames.9 There are results like this for other modal systems as well. The nomenclature will make it easy to remember the information in Table 2.3.

9 Soundness means that every theorem of $K$ is K-valid. Completeness means that every K-valid sentence is a theorem of $K$. Together these imply that $K = \text{the set of K-valid sentences}$. 
For example, T is sound and complete with respect to the set of reflexive frames; in other words, the set of MLL sentences which can be proven in the logic defined by the K and T axioms is equal to the set of T-valid MLL sentences. Note that there are also incomplete modal logics, logics which do not prove exactly the set of sentences which are valid on a class of frames which we can independently define; see a modal logic textbook for details (e.g. Hughes and Cresswell 1996: Ch. 9).

### 2.3 A linguistically realistic version of modal logic

Modal logic provides many important insights into the meanings of modal expressions, but as has been emphasized several times, its goal is not to provide a semantic analysis of natural language. Semanticists and logicians have different goals. When it comes to modality, the primary goal of the semanticist is to provide a precise theory of the meanings of modal expressions across languages which yields an accurate description of the facts and an explanation of linguistically important generalizations. The goal of the logician is to systematize and understand important features of reasoning with the concepts of necessity, obligation, and so forth. Because of this difference, modal logic ignores many important features of the meanings of modal expressions which are important to linguists. For example, it ignores the fact that in some languages epistemic modals occurs in a different position from deontic modals (see e.g. Cinque 1999). Just as importantly, modal logic does not integrate its ideas about the meanings of modal expressions into a general theory of natural language. Though sometimes the relationship between modal meanings and other meanings is discussed (for example, the relationship between modality and tense; see Section 5.1 and Thomason 2002 for a survey of work in logic), modal logic does not attempt to do this in a comprehensive way.

Though semanticists and logicians don't have the same goals, perhaps semanticists can just modify modal logic so as to make it fit with their goals. That is, can we not employ the same theoretical ideas (possible worlds, frames, etc.) with the purpose of providing a precise theory of the semantics of modal expressions across languages which yields an accurate description of the facts and an explanation of linguistically important generalizations? In this section, we'll see how far such a conception can take us. At first, we'll stay pretty close to the
ideas presented in Section 2.2, and then, bit by bit, we’ll add a number of refinements.

2.3.1 The Simple Modal Logic Hypothesis

We must stop working with □ and ◇ and put real words into the language under analysis. We are going to be interested in sentences like the following:

(35) (a) Must p.
(b) Should p.
(c) May p.
(d) Can p.

For the time being, we’ll assume that p can be any non-modalized, non-tensed English sentence, and that an appropriate syntactic analysis will place the modal in the right position. Therefore, (35a) could stand for sentences like the following:

(36) It must be raining outside.
    = must (it be raining outside.)

(37) Dog owners must keep their animals indoors.
    = must (dog owners keep their animals indoors.)

Among the patterns in (35), the first two contain modal operators which correspond to □. They may be called necessity modals. The second two, possibility modals, correspond to ◇. However, must and should are not identical, nor are may and can. For example, both may and can can be used deontically; may can be used epistemically, but can cannot (unless it’s negated):

(38) (a) Dogs may stay in this hotel. (deontic)
    (b) Dogs can stay in this hotel. (deontic)

(39) (a) It may be raining. (epistemic)
        #It can be raining. (epistemic)
    (b) It can’t be raining. (epistemic)

Likewise, must can be used both both epistemically and deontically. Should can clearly be used deontically, and while it can be epistemic as well, this use is less natural for many speakers.

Not only are there differences between epistemic and deontic modals; there seem to be sub-varieties of these categories as well. For example, though both may and can have deontic uses, as we can see
In (38) these deontic uses feel quite different. We need to develop a framework for describing and distinguishing all of the subtle and not-so-subtle differences among modal meanings.

In principle, modal logic gives us four things to work with in distinguishing the meanings of modals:

1. Whether the modal is a necessity modal (a kind of $\Box$) or a possibility modal (a kind of $\Diamond$).
2. The set $W$ of possible worlds.
3. The accessibility relation $R$.
4. The valuation function $V$.

When semanticists apply modal logic, or something close to it, to natural language, they typically work only with points 1 and 3. The idea of point 2, that $W$ could change, has something in common with Kratzer’s theory to be discussed in Section 3.1. As for point 4, the idea that a modal could depend on $V$, I don’t know of any proposal along these lines. Such a proposal would say that a modal can place a requirement on which atomic sentences are true, for example saying that a particular atomic sentence (e.g., *snow is white*) is true.\footnote{Some versions of dynamic logic have an operator $\hat{\phi}$, for each formula $\phi$. ($\hat{\phi}$ is a kind of $\Diamond$, and so we could also notate it as $\Diamond\hat{\phi}$.) The accessibility relation $R$ for $\hat{\phi}$ has $R(w,v)$ iff $\phi$ is true at $w$, $M$ and $w = v$. Note how the meaning of $\hat{\phi}$ depends whether $\phi$ is true, and this in turn can depend on the valuation. See Blackburn et al. (2001: 13, 23) for discussion. Though this operator might be useful for the analysis of some aspects of natural language (e.g. presupposition, cf. van Eijck 1996), it doesn’t seem to capture the interpretation of any modal element in natural language. For this reason I don’t discuss it any further here, since the present work is about the semantics of modality (more or less as traditionally defined) in natural language, not the applications of modal logic to natural language.}

It is worth making explicit the idea that the meanings of modal expressions in natural language can be analyzed in terms of points 1 and 3. We can call it the **simple modal logic hypothesis**:

\begin{itemize}
  \item [\textbf{40}] **Simple Modal Logic Hypothesis**: The meaning of every modal expression in natural language can be expressed in terms of only two properties:
    \begin{itemize}
      \item [\textbf{a}] Whether it is a necessity or a possibility modal, and
      \item [\textbf{b}] Its accessibility relation, $R$.
    \end{itemize}
\end{itemize}

The idea of this hypothesis is that the modal part of a given natural language is like MLL but with a number of $\Box$’s and $\Diamond$’s. We call it the **Simple Modal Logic Hypothesis** because we are leaving aside at this
point the possibility that the modal logic on which we build our analysis of natural language is something more sophisticated than MLL. We will discuss a theory which says that the basis should a different type of modal logic when we examine dynamic logic in Section 3.2.

2.3.2 Necessity and possibility

The following lists indicates which of the English modal auxiliaries correspond to □ and which correspond to ◁.

(41) (a) Necessity modals (□): must, should, would, (will, shall).
(b) Possibility modals (◁): may, might, can, could.

Sometimes people have the intuition that there should be more options than just □ and ◁. For example, deontic should seems to be weaker than deontic must, but certainly closer to must than to may:

(42) (a) You must leave right away.
(b) You should leave right away.
(c) You may leave right away.

However, it’s not clear how to define a concept of weak necessity, as contrasting with regular strong necessity, in a way that makes sense within the basic framework of modal logic. A common intuition is that we are talking not about all accessible worlds or some accessible world, but about most accessible worlds. Suppose that △ were such an operator.

(43) \[ \llbracket \triangle \beta \rrbracket^w_M = 1 \text{ iff for most } v \text{ such that } R(w, v), \llbracket \beta \rrbracket^v_M = 1 \]

What does it mean to talk about most worlds? If the set of worlds in question were finite, then it would be relatively clear. Suppose that there are 1,000 accessible worlds (worlds v such that R(w, v)). Then if 900 of them made β true, we’d have a ratio of 9:1 β worlds to non-β worlds, and we’d be satisfied that △β is true; and if only 100 of them made β true, we’d have a 1:9 ratio and be sure that △β is false. Somewhere around 500–600, it might not be so clear, but perhaps we could live with the vagueness.

The problem is that it is standardly assumed that the set of worlds W we are working with is infinite. W is the set of all possible worlds—every way the world could be—or perhaps an infinite subset of this like

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11 It is controversial whether will and shall should be in this table. This point will be discussed in Section 5.1.
all of the possible worlds which are conceivable by humans. (To see that
humans can conceive of an infinite number of possible worlds, consider
first the actual world, then the actual world with one additional star in
the universe, then the actual world with two additional stars, etc.) If
\( W \) is infinite, then typically the set of accessible worlds will be infinite
as well. Suppose that \( R \) is a deontic accessibility relation, and that \( v \) is
some world accessible from the actual world. It’s not relevant to any
system of rules I’m aware of how many stars are in the universe, and
so the world just like \( v \) except that it has one additional star is also
accessible; the same goes for two stars, and so forth. Therefore the set
of accessible worlds is infinite.

Since the set of accessible worlds is infinite, then typically the set of
accessible worlds in which \( \beta \) is true will be infinite (unless it’s empty).
For example, suppose that \( \beta \) is You leave right away. Take some acces-
sible world in which you leave right away. Then the worlds just like
this one except that there are one, two, three, etc., additional stars in
the universe are also accessible worlds in which you leave right away.
Therefore, to know whether (43) is true, we’re going to have to compare
two infinite sets: the set of accessible worlds in which you don’t leave
right away, and the set of accessible worlds in which you leave right
away. We are supposed to judge whether the ratio \( \infty:\infty \) is “most.” In
this case, our simple minded math for “most” breaks down. One would
have to seek a more sophisticated understanding “most” as it’s used in
(43) which takes into account infinite sets; as far as I know, this cannot
be done in a way that gives the correct meaning for words like should.

2.3.3 Accessibility relations

According to the Simple Modal Logic Hypothesis, the only way we
have at our disposal to distinguish the meanings of modals, other than
classifying them according to whether they correspond to \( \Box \) or \( \Diamond \), is by
assigning them different accessibility relations. So far, we have defined
epistemic and deontic accessibility relations. At this point, we need both
further kinds of accessibility relations and subtypes of the kinds we
already have.

The difference in strength between must and should might be ana-
lyzed by saying that they use different subtypes of deontic accessibility
relations. For example, following Bybee et al. (1994) we can distinguish
sets of rules depending on how serious the consequences are for not
following those rules. Given such a distinction, we might suggest that
while \textit{must} uses an accessibility relation based on a set of rules backed up by serious consequences, \textit{should} uses an accessibility relation based on a wider set of rules, including both rules which are backed up by potentially serious consequences and those which might be violated without anything very terrible happening. On this view, the following sentences are expected to be natural, given the kinds of accessibility relations associated with \textit{must} and \textit{should}:

(44) (a) I’m a month late in returning the semantics students’ assignments. I must grade them this weekend (potential negative impact if I don’t: the semantics students are very upset).

(b) I’m also two days late in returning the syntax students’ assignments. I should grade them this weekend as well (potential negative impact: the syntax students are somewhat upset).

If we assume that the set of rules which form the basis of \textit{should}'s accessibility relation includes all of those which form the basis of \textit{must}'s, then it follows that \textit{must} \( p \) entails \textit{should} \( p \); in other words, \textit{must} \( p \) is stronger than \textit{should} \( p \). This is so for the following reason. Let’s call the set of worlds accessible (from world \( w \)) under \textit{must}'s accessibility relation \( R_{\text{must}}(w) \) and that accessible under \textit{should}'s, \( R_{\text{should}}(w) \). Suppose that \( r_{\text{must}} \) is the set of rules on which \( R_{\text{must}}(w) \) is based and \( r_{\text{should}} \) is the set on which \( R_{\text{should}}(w) \) is based. The analysis we are considering says that \( r_{\text{must}} \) is a subset of \( r_{\text{should}} \). Notice that this implies that \( R_{\text{should}}(w) \) is a subset of \( R_{\text{must}}(w) \). That is, the fact that \textit{should} cares about more rules implies that it accesses fewer worlds.

The fact that if \( r_{\text{must}} \) is a subset of \( r_{\text{should}} \) implies that \( R_{\text{should}}(w) \) is a subset of \( R_{\text{must}}(w) \) exemplifies an important pattern, and so at this point it’s important to make clear why it holds. What we want to see is that, for any sets of sets \( M \) and \( S \), if \( M \subseteq S \), then \( \bigcap S \subseteq \bigcap M \). The subset relation reverses once we take intersections. Figure 2.2 illustrates a very simple example. In this figure, \( M \) contains the two sets drawn as ellipses with solid lines at the bottom, and \( S \) contains all three sets, the two indicated with solid lines plus the one drawn with a dashed line. Hence, \( \bigcap S \) is the dark gray area, while \( \bigcap M \) is the area which is either light or dark gray. Clearly, the dark gray area is a subset of the dark-or-light gray area.

Now we can see why \textit{must} \( p \) entails \textit{should} \( p \). The reason can be seen in Figure 2.3, a Venn diagram where the ellipses indicate sets of
possible worlds and \( w \) an arbitrary world which we count as the actual world. Given that the set of worlds accessed by \textit{should} is a subset of the set accessed by \textit{must}, if the latter is a subset of \( p \) (if \textit{must} \( p \) is true), clearly the former is also a subset of \( p \) (then \textit{should} \( p \) is also true). We’ll examine other ways of explaining the difference between \textit{must} and \textit{should} in Sections 3.1.3 and 4.3.3.

The difference between \textit{must} and \textit{should} shows that we need at least two kinds of deontic accessibility relations. But in fact we need even more. Consider the following uses of deontic \textit{must}:

\begin{enumerate}
\item[(45)] In view of the laws of Massachusetts, drivers \textit{must} yield to pedestrians.
\item[(46)] In view of the traditions of our family, you, as the youngest child, \textit{must} read the story on Christmas eve.
\end{enumerate}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2_2.png}
\caption{Intersection reverses the subset relation}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2_3.png}
\caption{Deontic \textit{Must} \( p \) entails deontic \textit{should} \( p \)}
\end{figure}
In view of the rules of student-teacher relationships, you must not yell at your teachers.

As pointed out by Kratzer (1977), we can multiply examples like these indefinitely. It seems that a given modal is compatible with many different accessibility relations, perhaps infinitely many. Therefore, when we are talking about "deontic must," we are not talking about something equivalent to a □ with a particular deontic accessibility relation we can specify. (One could imagine such a word existing, though it seems that no natural language has one.) Rather, we are talking about a word which is compatible with a range of accessibility relations, all of which can be classified as deontic.

Not all modality in natural language can be classified as epistemic or deontic. We need other categories. Some of the other general varieties of modality, besides epistemic and deontic, include modalities of ability, desire (called "bouletic modality"), goals (also called "teleological modality"), and perhaps history:

Fish can breathe water. (ability)

Given how you love chocolate, you should try this cake. (desire)

To get to my house, you must take a ferry. (goal)

Humanity may eventually destroy itself. (history)

We need accessibility relations corresponding to each of these types of modal. Here’s a suitable definition for the desire case:

Bouletic accessibility relation

R is a bouletic accessibility relation iff for some individual i, R = the relation which holds between two worlds w and w' iff all of i's desires in w are satisfied in w'.

A bouletic frame is one which has the set of possible worlds conceivable by humans as W and a bouletic accessibility relation as R. Assuming that should in (49) is a □ which uses a bouletic frame and that i is the addressee ("you"), the meaning of (49) is then as follows:

[ [ You should try this cake ] ] w.M = 1 iff the addressee tries this cake in every world in which his or her desires in w are satisfied.

For example, the addressee loves chocolate. This cake was made with good chocolate, and eating good chocolate always makes him happy. The cake is right in front of him, and nobody is going to get angry if he
takes a piece. If all of this is true, it is reasonable to say, on a bouletic interpretation, that (49) is true. (53) seems to capture the intuitive reason why it is true: the bouletically accessible worlds are those in which his desires are satisfied, and all such worlds are ones in which he tries the cake.

Next let’s examine the type of modality in (51). Some authors (e.g., Condoravdi 2002 and Werner 2006, see Sections 5.1.1–5.1.2) think that there is a kind of historical modality having to do with what the future might be like, given the history which has transpired so far. This is a kind of blended temporal-metaphysical modality based on the intuition that the past is fixed in a way that the future is not. For example, assuming that (51) represents historical modality, it means that everything which has happened so far does not rule out that humanity eventually destroys itself.

The accessibility relation needed to explain historical modality is developed in Kamp (1979) and Thomason (2002). It must have the properties outlined in (54):

(54) **Historical accessibility relation**

R is a historical accessibility relation iff for some time t, R = the relation which holds between two worlds w and w’ iff w and w’ are identical at all times up to and including t.

The central idea here is that historical modality assumes connection between the structure of time and the set of possible worlds, so that at any time, we can identify equivalence classes of worlds which are alike up through that time (and which only develop differences after that time). For any world, the accessible worlds are just those in its equivalence class. Notice that this definition implies that □p (on a historical interpretation) entails p, since a historical accessibility relation is reflexive.

It’s not easy to find a natural example with a clear historical interpretation. For example, (51) could be classified as epistemic rather than historical. Perhaps (55a) and (55b) are examples.

(55) (a) All life must someday end.

(b) He might have won the game. (Condoravdi (2002, (6b)))

12 The historical accessibility relation can be represented in terms of the “branching time” model of the relationship between time and modality. According to this view, time is not a line, but has a tree structure, with a fixed past (towards the root), but possibly many open futures (towards the leaves); see Section 5.1.1 for further discussion and Thomason (2002) for details.
I’m not entirely convinced by (55a); it doesn’t really seem to say that, given how history has developed so far, all life will someday end. That would leave open the possibility that, if history had developed somewhat differently, we would not be in this predicament. Rather, (55a) seems to report a timeless truth. As such, it would be an example of metaphysical modality, rather than historical modality. Condoravdi’s (55b) is the best candidate. This example implies that he did not win the game, and so if we assume that epistemic modals are always based on the speaker’s knowledge at the speech time, the reading cannot be epistemic. (It’s a delicate matter whether we should make this assumption about epistemic modals, as we’ll see below, especially Section 5.1.1.) It is possible that some sentences with will express historical modality, but as we’ll see in Sections 5.1.2, the analysis of will as a modal is controversial.

This discussion of varieties of deontic, bouletic, and historical modality should make it clear how we can think about and analyze varieties of modality within the Simple Modal Logic Hypothesis. Modal logic clearly provides tools which allow subtle and interesting explanations of the semantics of modal elements.

2.3.4 Problems with the Simple Modal Logic Hypothesis

Let us summarize our conclusions in this section so far. For every modal expression M:

1. Force: M is classified as either a necessity or possibility operator.
2. Accessibility Relations:
   (a) M is associated with some set \( A^M \) of accessibility relations.
   (b) These accessibility relations fall into one or more of the classes epistemic, deontic, bouletic, ability, historical, etc.
3. Meanings:
   (a) If M is a necessity operator, for each accessibility relation \( R \in A^M \), a sentence of the form \( M(\beta) \) can be interpreted as \( \Box \beta \) using the frame \( \langle W, R \rangle \).
   (b) If M is a possibility operator, for each accessibility relation \( R \in A^M \), a sentence of the form \( M(\beta) \) can be interpreted as \( \Diamond \beta \) using the frame \( \langle W, R \rangle \).

This group of ideas conforms to the Simple Modal Logic Hypothesis, and it does a pretty good job of capturing the meanings of sentences with modal auxiliaries. However, as a linguistic analysis, it has a very serious shortcoming: it massively over-generates. That is, it may allow
us to express every meaning a given modal sentence has, but it also leads us to expect very many meanings it doesn’t have. Let us see this point by examining epistemic must. Clearly, must has an epistemic interpretation; for example, suppose John is expecting Mary to stop by his house, but she’s not there yet. John can say (56):

(56) Mary must be lost.

In such a case, (56) is interpreted as a □ with the accessibility relation $R_{\text{John}}$:

(57) $R_{\text{John}}(w, w')$ iff everything John knows in $w$ is true in $w'$.

Now suppose that Joan is also expecting Mary, and that Mary is not at Joan’s house yet either. Joan also says (56). There the sentence is interpreted with the accessibility relation $R_{\text{Joan}}$:

(58) $R_{\text{Joan}}(w, w')$ iff everything Joan knows in $w$ is true in $w'$.

So far so good. However, the theory also predicts that when John says (56), it can be interpreted with respect to $R_{\text{Joan}}$, and when Joan says it, it can be interpreted with respect to $R_{\text{John}}$. That is, John should be able to mean “Joan’s knowledge implies that Mary is lost,” and Joan should be able to mean “John’s knowledge implies that Mary is lost.” Clearly, these are not real options. When John speaks, it’s about his knowledge, and when Joan speaks, it’s about hers. Our theory as it stands does not allow us to rule out the incorrect interpretations.

A similar problem arises once consider the role of time in epistemic modality. We cannot define an epistemic accessibility relation simply in terms of what some individual $i$ knows. What we know changes over time, and so we we really have to define epistemic accessibility relations in terms of what $i$ knows at $t$:

(59) **Epistemic accessibility relation**

  $R$ is an epistemic accessibility relation iff, for some individual $i$ and some time $t$, $R = \text{the relation which holds between two worlds } w \text{ and } w' \text{ iff everything which } i \text{ knows at } t \text{ in } w \text{ is also true in } w'$.

The accessibility relation $R_{\text{John–on–Tuesday}}$ which expresses what John knows on Tuesday is different from that one $R_{\text{John–on–Friday}}$ which expresses what he knows on Friday. If he says (56) on Tuesday, the meaning is based on $R_{\text{John–on–Tuesday}}$, and if he says it on Friday, the meaning is based on $R_{\text{John–on–Friday}}$. When he says it on Tuesday, the meaning
cannot be based on _RJohn-on-Friday_, and vice versa. That is, the time \( t \) has to be the time at which the sentence is spoken. This fact is not captured by the Simple Modal Logic Hypothesis as summarized in 1–3 above.

Analogous problems can be constructed for modals other than epistemic ones. Historical modality is a very clear case in point. The way (54) is set up, we have a different historical accessibility relation for each time \( t \). But when we use a historical modal (assuming there are any), we are not free to interpret it with any historical accessibility relation we want. Rather, the time in question must be the time at which the sentence is used, at least with simple examples like (51) and (55). It seems that some innovations in our semantic system are called for.

The general issue that emerges is that it is not enough to classify each modal as epistemic, deontic, etc. In determining which accessibility relation to use, we also need to make reference to such concepts as the identity of the speaker and the time at which the sentence is used. In what follows, we will see how to build such information into the semantics of modals in a way that the unattested interpretations of (56) can be ruled out. It should be noted that this enriched system can no longer be identified with the simple modal logic presented in Section 2.2. Nevertheless, it is still close enough that most semanticists would be willing to describe it as embodying an attempt to analyze English modal auxiliaries within modal logic.

### 2.3.5 The indexicality of modals

We have seen at least two things which can be relevant to choosing a correct accessibility relation for a modal, other than the classification of the relation as epistemic, deontic, etc. These are the speaker and the time at which the modal sentence is used. A striking thing about these two is that they are both _indexical_ concepts. (Some linguists would call them _deictic_ concepts.) Indexicals are elements in natural language whose meanings make essential reference to the situation in which they are used, the _context of utterance_. Simple indexical expressions include the words _I_, _now_, and _here_. The meaning of _I_, for example, is that it introduces reference to the speaker into the process of semantic composition. In the case of (60), the subject _I_ refers to Joan, and so the sentence is true iff Joan is happy.

(60)  Joan: I am happy.

Following Kaplan (1989), we can distinguish two “levels” of the meaning of an expression: its **character** and its **content**. The character of an expression is its context-independent meaning: the character of _I_ is
to pick out the speaker. The content of an expression is the contribution it makes to semantics within a particular context. The content of \( I \) in (60) is Joan. Character can be modeled formally as a function, at least in many cases. For example, we can think of the character of \( I \) as in (61a); then, if we call the context of Joan’s utterance in (60) \( u_j \), we calculate the content as in (61b):

\[
\text{(61) (a) The character of } I = \text{ the function } f_I \text{ from contexts to individuals such that, for any context } c, f_I(c) = \text{ the speaker in } c.
\]

\[
\text{(b) } f_I(u_j) = \text{Joan}
\]

When we have a sentence containing an epistemic modal, we want to make sure that the individual \( i \) and the time \( t \) referred to in the definition of the epistemic accessibility relation (59) are the speaker and the time at which the modal is used. That is, we want \( f_I(c) \) and \( f_{\text{now}}(c) \) to be used in the semantic rule used to interpret the modal. One way to accomplish this formally is in terms of an accessibility relation function:

\[
\text{(62) Accessibility relation function}
\]

\[
A \text{ is an accessibility relation function iff}
\]

1. Its domain is a set of actual and/or hypothetical contexts of utterance, and
2. Its range is a set of accessibility relations.

Just as we distinguish subtypes of accessibility relations, we can distinguish subtypes of accessibility relation functions. For example:

\[
\text{(63) Epistemic accessibility relation function}
\]

\[
A \text{ is an epistemic accessibility relation function iff}
\]

1. \( A \) is an accessibility relation function, and
2. For every context \( c \) in the domain of \( A \), \( A(c) = \text{the relation which holds between two worlds } w \text{ and } w' \text{ iff everything } f_I(c) \text{ knows at } f_{\text{now}}(c) \text{ in } w \text{ is also true in } w'.
\]

A slightly different way of incorporating indexicality into the semantics of modals is by making the accessibility relations themselves into relations between various aspects of the context and possible worlds. For example, we could build our semantics on the relation (64):

\[
\text{(64) } A \text{ is an epistemic accessibility relation iff for any worlds } w \text{ and } w', \text{ any individual } i \text{ and any time } t, A = \text{the relation which holds between } \langle w, i, t \rangle \text{ and } w' \text{ iff everything } i \text{ knows in } w \text{ at } t \text{ is true in } w'.
\]
In this formulation, the accessibility relation is not merely a relation between worlds. It is a relation between triplets of a world, individual and time, on the one hand, and worlds, on the other.

In order to account for the indexicality of modals, we also must revise the semantic rule used to interpret modal sentences. Previously, our interpretation function $[[w, M]]$ had two parameters: a world and a model. A model consisted of a frame (a pair of a set of worlds $W$ and an accessibility relation $R$) and a valuation function $V$. What we want to do is to change this a bit so that features of the context of utterance are available to the modal. Here’s one way to do this: A frame is now a pair $(W, A)$ of a set of worlds and an accessibility relation function. A model is a frame and a valuation. And the interpretation function has three parameters: a world, a context and a model. All of these changes are summarized in Table 2.4.13 (Let me warn you that the parameters on the semantic value function $[[ ]]$ will change several more times in this book.14 Each change reflects some new ideas concerning the the semantic theory of modality. Look on the changing parameters not as an annoyance, but rather as a convenient summary of the commitments of a particular theory.)

Now we are ready for a new semantic rule for $\Box$. In terms of the idea of accessibility relation functions, defined in (62), it can be given in (65). (A parallel new rule can be given for $\Diamond$ as well, but I won’t bother to present it explicitly.)

\[
\begin{align*}
\text{(65)} & \quad [[\Box \beta ]]_{w, c, ((W, A), V)} = 1 \text{ iff for all } v \text{ such that } A(c)(w, v), \\
& \quad [[\beta ]]_{v, c, ((W, A), V)} = 1
\end{align*}
\]

13 It is possible to define the various notions of validity in terms of the new system, and it would make a good exercise to write out the necessary rules in full detail.

14 Sometimes we “flatten” all of the angled brackets out, so that instead of $[[w, c, ((W, A), V)]]$, we just write $[[w, c, W, A, V]]$. And sometimes we don’t write $W$ and $V$ at all: $[[w, c, A]]$.
The definition in (65) gives the same basic meaning to \( \Box \) as in MLL, but instead of using an accessibility relation directly, it calculates one by applying the accessibility relation function \( A \) to the context of utterance \( c \). Different varieties of modals are interpreted with respect to different kinds of accessibility relation functions: An epistemic modal uses an epistemic accessibility relation function and a deontic modal uses a deontic accessibility relation function. Various subtle differences among modals (like that between \textit{must} and \textit{should}) will be explained in terms of the accessibility relation functions with respect to which they are interpreted.

One final improvement is necessary before we have can say that we have a version of modal logic which is as good as it’s going to get for the purposes of analyzing natural language. Right now, the model only has one accessibility relation function and one context. This means that if two modals occur in the same sentence, they will use the same accessibility relation. Any two modals in the sentence of the same force would therefore have to be synonymous. This is clearly not right:

(66)  It must be raining and so we should take an umbrella.

In (66), the first modal is epistemic, and the second is deontic. However, if we compute \([ [ (66) ] ]^{w,c,\langle W,A \rangle,V} \), and \( A(c) \) happens to be an epistemic accessibility relation, both modals will receive an epistemic interpretation, and if it happens to be a deontic accessibility relation, both modals will be interpreted as deontic. We need to make available a number of accessibility relations.

In principle, we could produce multiple accessibility relations in two ways. One would be to have multiple accessibility relation functions. On this view, a frame would have the structure \( \langle W, A_1, A_2, \ldots, A_n \rangle \), where \( n \) is the number of different kinds of accessibility relation functions needed for the language in question. (Example (66) shows that \( n \) is at least 2 for English.) The other way to get multiple accessibility relations would be to have multiple contexts, so that the context used when we assign a meaning to \textit{must} is different from the one used when we assign a meaning to \textit{should} in (66). If the contexts are different, a single \( A \) can assign different accessibility relations to the two contexts. Say the context for \textit{must} is \( c_1 \) and that for \textit{should} is \( c_2 \). Then \( A(c_1) \) would be epistemic and \( A(c_2) \) would be deontic. Kratzer (1978) develops the second option in a somewhat different framework, but here the first option is simpler.
Following the first option, then, a frame can be defined as a tuple consisting of a set of possible worlds and some number of accessibility relation functions. Each one of them is required to be of a particular kind. For example, we might say that the first has to be an epistemic, the second deontic of the sort used by should, the third bouletic, the fourth deontic of the sort needed by must, and so forth.

(67) An English frame is a tuple \( \langle W, A_1, A_2, \ldots, A_n \rangle \), where:
1. \( W \) is a set of possible worlds.
2. \( A_1 \) is an epistemic accessibility relation function.
3. \( A_2 \) is a deontic accessibility relation function which takes into account both rules which are backed up by serious consequences and rules which are not.

etc.

The interpretation function for English therefore now looks like this:

\[
[[\text{ }]]_{w,c,\langle\langle W, A_1, A_2, \ldots, A_n \rangle, V\rangle}
\]

Each modal expression is interpreted by a rule which makes reference to a particular \( A \). For example:

(68) (a) Must\( E \beta \) is interpreted as \( \square_1 \beta \), defined as follows:
(b) \( [[\square_1 \beta]]_{w,c,\langle\langle W, A_1, A_2, \ldots, A_n \rangle, V\rangle} = 1 \) \( \iff \)
for all \( v \) such that \( A_1(c)(w, v) \),
\( [[\beta]]_{v,c,\langle\langle W, A_1, A_2, \ldots, A_n \rangle, V\rangle} = 1 \)

(69) (a) Should\( D \beta \) is interpreted as \( \square_2 \beta \), defined as follows:
(b) \( [[\square_2 \beta]]_{w,c,\langle\langle W, A_1, A_2, \ldots, A_n \rangle, V\rangle} = 1 \) \( \iff \)
for all \( v \) such that \( A_2(c)(w, v) \),
\( [[\beta]]_{v,c,\langle\langle W, A_1, A_2, \ldots, A_n \rangle, V\rangle} = 1 \)

Here, epistemic must \( (= \text{must}_E) \) uses the first accessibility relation function, \( A_1 \), which by (67) has to be epistemic. All of the other accessibility relation functions are irrelevant to must\( E \). Deontic should uses the second, \( A_2 \), and along these lines we’d have additional rules to associate each modal expression with the correct member of the sequence \( A_1, A_2, \ldots, A_n \).

\( 15 \) Another option would be to have a series of frames in the modal: \( M = \langle\langle W_1, A_1 \rangle, \langle W_2, A_2 \rangle, \ldots, \langle W_n, A_n \rangle, V\rangle \). However, unless we can find a reason why the various sets of possible worlds might differ, this would be needlessly complex.
Table 2.5. Summary of differences between the traditional frame-based semantics of modal logic and the revised system

<table>
<thead>
<tr>
<th>Traditional Modal Logic: $\mathbb{L} \models \phi$ (W, R, V)</th>
<th>Modal Logic for Linguistics: $\mathbb{L} \models \phi$ (W, A₁, A₂, ..., Aₙ, V)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Frame</strong> A set of worlds W and an accessibility relation R</td>
<td>A set of worlds W and a series of n accessibility relation functions $A₁, A₂, \ldots, Aₙ$</td>
</tr>
<tr>
<td><strong>Model</strong> A frame and a valuation</td>
<td>A frame and a valuation</td>
</tr>
<tr>
<td><strong>Parameters</strong> A world and a model</td>
<td>A world, a context, and a model</td>
</tr>
</tbody>
</table>

2.3.6 Summary

Modal logic can analyze the meanings of modal expressions in natural language by categorizing each one as a version of $\Box$ or $\Diamond$ and by assigning each an appropriate accessibility relation. In order to associate a modal expression with the right accessibility relation, we make two important changes to the system of MLL. First, we made the accessibility relation depend on the context of utterance. This allows for indexical concepts like the speaker and time of utterance to assist in finding the right accessibility relation. And second, we expand the frame to include multiple accessibility relation functions, so that modals of different kinds can each be associated with the correct one.

2.4 Looking ahead

A basic understanding of modal logic is invaluable to linguists who study modality because some of the first theoretically precise ideas about the semantics of modal expressions were developed within modal logic. Most fundamentally, the notion of possible world allows us to represent directly the intuition that modality has to do with possible but not necessarily actual situations. The modal operators are quantifiers over possible worlds; in particular, they are universal (the $\Box$) or existential (the $\Diamond$) quantifiers with a domain of quantification picked out by an accessibility relation. Within this system, logicians have discovered that formal properties of accessibility relations, such as reflexivity, seriality, and so forth, correspond to interesting logical properties in the operators themselves. For example, seriality corresponds to the D axiom, and therefore is appropriate for deontic logic. Such correspondences have been inspiring to linguists, since they suggest that we may
be able to explain some of the semantic properties of natural language modals in a similar way.

Classical modal logic is not a semantic theory of natural language modals, but as discussed in Section 2.3 one can adapt the core ideas of modal logic for the purposes of linguistic analysis. However, few linguists believe that this adapted “Modal Logic for Linguists” provides an adequate linguistic analysis. In the next chapter, we will survey three approaches to the semantics of modality within linguistics, examining both the empirical phenomena which motivate them and the details of the theories themselves. All of these approaches diverge significantly from modal logic in how they explain the semantics of modal expressions, but some do so more than others. Kratzer’s theory, discussed in Section 3.1 is the most closely related to modal logic of the three. Dynamic semantics, presented in Section 3.2, uses the concept of possible world, but does not treat epistemic modals as quantifiers. Cognitive and functional approaches, of which I give an overview in Section 3.3, don’t use the notion of possible world at all.

As we look at these new semantic theories of modality, modal logic provides an important benchmark for evaluation. Are they as precise as modal logic? Do they represent a fundamental intuition about the semantics of modality as clear and compelling the key intuition of modal logic, namely, that modal statements are true or false based on alternative ways the world could be? Do they provide better empirical coverage than modal logic, or better explanations of linguistic phenomena? At the end of this discussion, we will see that some ideas drawn from modal logic remain essential to our understanding of modality in human language.