Modals & Conditionals
Possible-Worlds Semantics for Conditionals

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Notation

· $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ are variables for sentences
  · use pseudo-logical notation for connectives and operators, e.g., conjunction ($\mathcal{A}$ and $\mathcal{B}$), negation (not-$\mathcal{A}$), or modals (must $\mathcal{A}$)
  · $\mathcal{A} \supset \mathcal{C}$ is a stand-in for a conditional construction
    · $\mathcal{A}$ – antecedent
    · $\mathcal{C}$ – consequent
  · we use $\Rightarrow$ to indicate an inferential relation between sentences or sets/conjunctions thereof
· $A, B, C, \ldots$ are variables for the corresponding propositions (sets of possible worlds)
  · use set-theoretic operations, e.g., intersection $A \cap B$, or complement $\overline{C} = W \setminus C$
· $a, b, c, \ldots$ are the corresponding proposition letters, i.e., input to valuation functions to return either true or false
  · use logical notation, e.g., $a \land b$, $\neg a$ etc.
1 Some Desiderata for a Semantics of Conditionals

- if we have an analysis for a conditional \((A > C)\) then this will make certain inferences valid or not
  - we assume that the interpretation of other connectives (negation, disjunction, conjunction) remains classically Boolean
- to judge an analysis of a conditional as good or bad we check which inferences are licensed and which are not
  - this only makes sense if we can reach some pre-theoretic agreement on which inferences should be licensed and which should be invalid
    - we should keep in mind that there is always the possibility of delegating an explanation of validity or invalidity instead to pragmatics
    - still, it is *prima facie* reasonable to try to come “as close as possible” to matching the desired inference patterns already in the semantics
  - the list of “goods” and “bads” below is inspired by Egré and Cozic (2008)

1.1 The Good

- *modus ponens:*
  \[ A > C, \ A \implies C \]

- *modus tollens:*
  \[ A > C, \ \text{not-}C \implies \text{not-}A \]

- *import-export principle:*
  \[ (A \text{ and } B) > C \iff A > (B > C) \]

- *simplification of disjunctive antecedents:*
  \[ (A \text{ or } B) > C \iff A > C, \ B > C \]

- *disjunction-to-conditional:*
  \[ A \text{ or } C \implies \text{not-}A > C \]

1.2 The Bad

- *ex falso quod libet:*
  \[ \text{not-}A \implies A > C \]
  
  (1) a. The moon is not made of green cheese.
      b. ∴ If the moon is made of green cheese, then 4+4=9.

- *verum ex omni:*
  \[ C \implies A > C \]
  
  (2) a. Angela Merkel is presently chancellor.
b. \[ \therefore \text{ If the moon is made of green cheese, Angela Merkel is presently chancellor.} \]

- **contraposition:**
  \[ A > C \Rightarrow \text{ not-}C > \text{ not-}A \]

  (3) a. If the U.S. halts the bombing, then North Vietnam will not agree to negotiate.
  
  b. \[ \therefore \text{ If North Vietnam agrees to negotiate, then the U.S. will not have halted the bombing.} \]
  
  \[ \text{ (Stalnaker 1968, p. 107) } \]

  (4) a. You win if this coin comes up heads.
  
  b. \[ \therefore \text{ If you loose, this coin will have come up tails.} \]

- **transitivity:**
  \[ A > B, B > C \Rightarrow A > C \]

  (5) a. If J. Edgar Hoover had been born a Russian, the he would today be a communist.
  
  b. If J. Edgar Hoover were today a communist, then he would be a traitor.
  
  c. \[ \therefore \text{ If J. Edgar Hoover had been born a Russian, he would be a traitor.} \]
  
  \[ \text{ (Stalnaker 1968, p. 106) } \]

- **monotonicity / strengthening of the antecedent:**
  \[ A > C \Rightarrow (A \text{ and } B) > C \]

  (6) a. If you put sugar in your coffee, it will taste better.
  
  b. \[ \therefore \text{ If you put sugar and gasoline in your coffee, it will taste better.} \]

- **negation:**
  \[ \text{ not } (A > C) \Leftrightarrow A \text{ and not-}C \]

  (7) a. It’s not the case that I lose my job if I am late.
  
  b. \[ \therefore \text{ I am late and I do not lose my job.} \]

1.3 **The Ugly**

- inferential properties may be interdependent

  - transitivity implies monotonicity (if \( (A \text{ and } B) > A \) is always true)
  
  - with Boolean negation, disjunction and conjunction, simplification of disjunctive antecedents implies monotonicity:
    \[ \text{ (cf. Warmbröd 1981) } \]
    
    - assume \( A > C \)
    
    - for arbitrary \( B, A \) is equivalent to: \( (A \text{ and } B) \) or \( (A \text{ and not-}B) \)
    
    - so we substitute: \( ((A \text{ and } B) \text{ or } (A \text{ and not-}B)) > C \)
    
    - from simplification of disjunctive antecedents we get: \( (A \text{ and } B) > C \)
2 Material Implication

- A MATERIAL IMPLICATION ANALYSIS of conditionals assumes that the meaning of conditionals is sufficiently captured by the truth-conditions for material implication:

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<th>a → c</th>
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- this yields an analysis for conditionals in set-theoretic terms as:

\[
\llbracket A > C \rrbracket = \overline{A} \cup C
\]

2.1 Problems with a Material Implication Analysis

- renders all of the above “bad” inferences good

3 Variably-Strict Implication

- the denial of a conditional like (8a) plausibly has the form (8b)

\begin{align*}
\text{(8)} & \quad \text{a. A: If the butler did not do it, the gardener has.} \\
& \quad \text{b. B: No, it’s possible they both didn’t do it.}
\end{align*}

- that suggests that conditionals relate to modals

- taking this seriously we’d get (with \(\mathcal{M} = (W, R, V)\) a suitable modal model):

\[
\llbracket \text{not }(A > C) \rrbracket^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \Box(a \land \lnot c)\}
\]

from which we compute backwards:

\[
\llbracket A > C \rrbracket^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \not\models \Box(a \land \lnot c)\}
\]

\[
= \{w \in W \mid \mathcal{M}, w \models \neg \Box(a \land \lnot c)\}
\]

\[
= \{w \in W \mid \mathcal{M}, w \models \Box(\lnot a \lor c)\}
\]

\[
= \{w \in W \mid \mathcal{M}, w \models \Box(a \rightarrow c)\}
\]

- thus, a VARIABLY-STRICIT IMPPLICATION ANALYSIS of conditionals assumes a contextually specified \(\mathcal{M}\) for which:

\[
\llbracket A > C \rrbracket^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \Box(a \rightarrow c)\}
\]

3.1 Advantages of a Variably-Strict Implication Analysis

- conservative extension of standard concepts:

  - subsumes a material implication analysis as a special case where \(wRw'\) iff \(w = w'\)
  
  - subsumes so-called strict implication as a special case where \(wRw'\) for all \(w, w'\)

  - fares better with the ex falso-problem (but see also below):
· we don’t get that sentences like (1b) are necessarily true if the antecedent is false in the actual world
· makes transparent a connection between modals and conditionals
· it seems that the modal interpretation of a conditional may depend on a modal in its consequent:

(9) deontic:
   a. If you caused the mess, then you have to clean it up too.
   b. No, it’s permissible for me to cause mess and not clean it up.
(10) future-will:
   a. If the weather is nice, we will have a barbecue.
   b. No, it might turn out that the weather is fine and we don’t have a barbecue.

3.2 Problems with a Variably-Strict Implication Analysis
· still suffers from a milder variant of the ex falso-problem:
  · any conditional with a logically false antecedent, such as \((\mathcal{A} \text{ and not-} \mathcal{A}) > C\) is still always true
· still suffers from monotonicity, so (6a) is still predicted to entail (6b)
· the analysis is not surface-compositional and depends on (something like) quantifier raising:

\[
\mathcal{A} > \text{must}(C) \sim \text{must}(\mathcal{A} > C)
\]
· quantifier raising does not work for conditionals with non-universal operators: (cf. Lewis 1975)

(11) a. If the weather is fine, we might have a barbecue.  ≠ might(\mathcal{A} > C)
b. If the weather is fine, we often have a barbecue.  ≠ often(\mathcal{A} > C)
c. If a man owns a donkey, he usually beats it.  ≠ usually(\mathcal{A} > C)

4 If-Clauses as Domain Restrictors
  · cases like (11) suggest that we should think of a conditional’s antecedent as the restrictor of the domain of some quantifier Q whose scope is the consequent (Lewis 1975); schematically:

\[
\mathcal{A} > Q(C) \sim Q_{\mathcal{A}}(C)
\]
  · we are especially concerned with cases where Q is either a universal or existential modal (cf. Kratzer 1991)
  · if there is no appropriate overt quantifier, assume a covert universal modal of some appropriate kind
  · standard modal logic does not have the necessary generalized modal quantifiers, but it is easy to introduce them:

\[
\begin{align*}
\mathbb{M}, w \models \Box(p, q) & \iff \text{for all } w' \in R(w), \text{ if } \mathbb{M}, w' \models a \text{ then } \mathbb{M}, w' \models c \\
\mathbb{M}, w \models \Diamond(p, q) & \iff \text{there is a } w' \in R(w) \text{ such that } \mathbb{M}, w' \models a \text{ and } \mathbb{M}, w' \models c
\end{align*}
\]
  · this gives the analysis:

\[
\begin{align*}
\text{\lbrack } \mathcal{A} > \text{must}(C) \rbrack^\mathbb{M} &= \{ w \in W \mid \mathbb{M}, w \models \Box(a, c) \} \\
&= \{ w \in W \mid R(w) \cap A \subseteq C \} \\
\text{\lbrack } \mathcal{A} > \text{might}(C) \rbrack^\mathbb{M} &= \{ w \in W \mid \mathbb{M}, w \models \Diamond(a, c) \} \\
&= \{ w \in W \mid (R(w) \cap A) \cap C \neq \emptyset \}
\end{align*}
\]
4.1 Problems with this Analysis

- still suffers from the logical ex falso-problem
- still suffers from monotonicity
- we predict \((A > \text{might}(C))\) and \((C > \text{might}(A))\) to be equivalent, but witness:

\[(12)\]

a. If the weather is nice, we might go swimming.
b. If we go swimming, the weather might be nice.

5 Order-Sensitive Implication

- to overcome also these last problems, we assume that the evaluation of modals takes ordering information into account (e.g. Stalnaker 1968; Lewis 1973; Stalnaker 1975; Veltman 1985; Kratzer 1991)
- consider enriched modal models:

\[\mathfrak{M} = (W, R, V, \{\leq_w\})\]

where \((W, R, V)\) is a modal model as before and \(\leq_w\) is an ordering on \(R(w)\) for each \(w \in W\)
- to facilitate notation we assume that all \(\leq_w\) are well-founded (Lewis 1973, “limit assumption”):

\[
\begin{align*}
&\text{centering: for all } u, v \in R(w), u \leq_w v \text{ or } v \leq_w u \\
&\text{comparability: for all } u, v \in R(w), u \prec_w v \text{ or } v \prec_w u \\
&\text{totality: for all } u, v \in R(w), u \neq v, then u <_w v \text{ or } v <_w u \\
&\text{this gives the following rough options (with imprecise appellation):}
\end{align*}
\]

Stalnakerian: all \(\leq_w\) satisfy centering, comparability, totality, and transitivity
- think: map each accessible world to a unique number for comparison

Lewisian: all \(\leq_w\) satisfy centering, comparability, and transitivity
- think: map each accessible world to a non-unique number for comparison
- think: a system of spheres

Premisse-Based: all \(\leq_w\) satisfy centering, and transitivity

\[
\text{(roughly Veltman 1985; Kratzer 1991)}
\]

\[
\begin{align*}
&\text{S}_\text{talnakerian}: \text{all } \leq_w \text{ satisfy centering, comparability, totality, and transitivity} \\
&\text{Lewisian}: \text{all } \leq_w \text{ satisfy centering, comparability, and transitivity} \\
&\text{Premisse-Based: all } \leq_w \text{ satisfy centering, and transitivity} \\
&\text{allow for incomparability}
\end{align*}
\]
5.2 Remarks on Some Relevant Properties

- We finally solve the problem of monotonicity

- A Stalnakerian semantics makes no difference between “must” and “might” conditionals: there is always only one minimal world

- A Stalnakerian semantics also makes the following principle true:

  Conditional excluded middle: it is always the case that either \( A > C \) or \( A > \neg C \) is true

Exercises

1. Which of the “good” and “bad” inferences do you accept as so classified? Where do you see options for a pragmatic explanation?

2. Calculate for each “good” and “bad” example from Section 1 whether (i) material, (ii) variably-strict and (iii) any variety of order-sensitive implication validates these inferences.

3. Show that a Stalnakerian semantics makes “conditional excluded middle” true, but not a Lewisian.

4. What do you think would incomparability in a conditional semantics, as with the “premisse-based” approach be needed for?

References


